Chapter - 1

Vector Spaces

Vector Space

Let $(F, +;)$ be a field. Let V be a non empty set whose elements are vectors. Then V is a vector space over the field F, if the following conditions are satisfied:

1. (*V*, +) is an abelian group

(i) ***Closure property***: V is closed with respect to addition i.e.,

$$\alpha \in V, \beta \in V \Rightarrow \alpha + \beta \in V$$

(ii) ***Associative***: $\alpha + (\beta + \Upsilon) = (\alpha + \beta) + \Upsilon, \forall \alpha, \beta, \Upsilon \in V$

(iii) ***Existence of identity***: $\exists$ an elements $0 \in V$ (zero vector) such that

$$\alpha + 0 = \alpha, \forall \alpha \in V$$

(iv) ***Existence of inverse***: To every vector α in V can be associated with a unique vector $-\alpha$ in V called the additive inverse *i.e.,*

$$\alpha + (-\alpha) = 0$$

(v) **Commutative:** $\alpha + \beta = \beta + \alpha, \forall \alpha, \beta \in V$

2. *V* is closed under scalar multiplication *i.e.,*

$$a \in F, \alpha \in V \Rightarrow a\alpha \in V$$

3. Multiplication and addition of vector is a distributive property i.e.,

(i) $a(\alpha + \beta) = a\alpha + a\beta, \forall a \in F, \alpha, \beta \in V$

(ii) $(a + b)\alpha = a\alpha + b\alpha, \forall a, b \in F, \alpha \in V$

(iii) $(ab)\alpha = a(b\alpha), \forall a, b \in F, \alpha \in V$

(iv) $1 \cdot \alpha = \alpha, \forall \alpha \in V$ and 1 is the unity element in F.

Example 1. The vector space of all ordered n-tuples over a field *F*.

Proof. Let *F* be a field. An ordered set $\alpha = (a_1, a_2, \ldots.a_n)$ of n-elements in *F* is called an n-tuples over *F*. Let*V* be the all ordered *n*-tuple over *F*. *Let* $V = \{(a_1, a_2, \ldots.a_n) : a_1, a_2, \ldots.a_n \in F\}$. Now, we will prove that *V* is a vector space over the field *F*. For this we define two n-tuples, addition and multiplication of two n-tuples by a scalar as follows.

Equality of two n-tuples : Let $\alpha = (a_1, a_2, \ldots.a_n)$ and $\beta = (b_1, b_2, \ldots.b_n)$ of *V*. Then $(a_1, a_2, \ldots.a_n) = (b_1, b_2, \ldots.b_n) \Rightarrow a_i = b_i, \forall i = 1, 2, \ldots, n$.

Addition of n-tuples : we take

$$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n), \forall \alpha = (a_1, a_2, \ldots.a_n) \in V,$$

$$\beta = (b_1, b_2, \ldots.b_n) \in V$$

Since $a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n$ are all elements of *F*, therefore, $\alpha + \beta \in V$ and thus *V* is closed with respect to addition of n-tuples. Scalar multiplication of n-tuples : we define.

$$a\alpha = (aa_1, aa_2, \ldots, aa_n), \forall a \in F, \alpha = (a_1, a_2, \ldots.a_n) \in V,$$

Since $aa_1, aa_2, \ldots, aa_n$ are all elements of *F*, therefore $a\alpha \in V$ and thus *V* is closed w.r.t. multiplication of n-tuples.

Now, we shall show that V is a vector space for the above two compositions.

1. (i) Associative : Let $(c_1, c_2, \ldots, c_n) = \Upsilon \in V$

$$\begin{aligned}
\alpha + (\beta + \Upsilon) &= (a_1, a_2, \ldots, a_n) + [(b_1, b_2, \ldots, b_n) + (c_1, c_2, \ldots, c_n)] \\
&= (a_1, a_2, \ldots, a_n) + [b_1 + c_1, b_2 + c_2, \ldots, b_n + c_n] \\
&= a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots, a_n + (b_n + c_n) \\
&= (a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots, (a_n + b_n) + c_n \\
&= [(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)] + (c_1, c_2, \ldots, c_n) \\
&= (\alpha + \beta) + \Upsilon
\end{aligned}$$

(ii) Commutative: We have

$$\begin{aligned}
\alpha + \beta &= (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) \\
&= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \\
&= (b_1 + a_1, b_2 + a_2, \ldots, b_n + a_n) \\
&= (b_1, b_2, \ldots, b_n) + (a_1, a_2, \ldots, a_n) \\
&= \beta + \alpha
\end{aligned}$$

(iii) Existence of Identify : Let $(0, 0, \ldots, 0) \in V$ then, we have

$$\begin{aligned}
\alpha + 0 &= (a_1, a_2, \ldots, a_n) + (0, 0, \ldots, 0) \\
&= (a_1 + 0, a_2 + 0, \ldots, a_n + 0) \\
&= (a_1, a_2, \ldots, a_n) = \alpha
\end{aligned}$$

(iv) Existence of Inverse : If $\alpha = (a_1, a_2, \ldots, a_n)$ then

$$-\alpha = (-a_1, -a_2, \ldots, -a_n) \in V$$

Then we have

$$\begin{aligned}\alpha + (-\alpha) &= (a_1, a_2, \ldots, a_n) + (-a_1, -a_2, \ldots, -a_n)\\ &= (a_1 - a_1, a_2 - a_2, \ldots, a_n - a_n)\\ &= (0, 0, \ldots, 0)\end{aligned}$$

Hence V is an abelian group under addition.

2. (i) If $a \in F$ and $(a_1, a_2, \ldots, a_n) = \alpha \in V$, $(b_1, b_2, \ldots, b_n) = \beta \in V$ then,

$$\begin{aligned}a(\alpha + \beta) &= a[(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)]\\ &= a[a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n]\\ &= a(a_1 + b_1), a(a_2 + b_2), \ldots, a(a_n + b_n)\\ &= (aa_1 + ab_1, aa_2 + ab_2, \ldots, aa_n + ab_n)\\ &= (aa_1, aa_2, \ldots, aa_n) + (ab_1, ab_2, \ldots, ab_n)\\ &= a(a_1, a_2, \ldots, a_n) + a(b_1, b_2, \ldots, b_n) = a\alpha + a\beta\end{aligned}$$

(ii) If $a, b \in F$ and $\alpha = (a_1, a_2, \ldots, a_n) \in V$ then

$$\begin{aligned}(a + b)\alpha &= (a + b)(a_1, a_2, \ldots, a_n)\\ &= [(a + b)a_1, (a + b)a_2, \ldots, (a + b)a_n]\\ &= (aa_1 + ba_1, aa_2 + ba_2, \ldots, aa_n + ba_n)\\ &= (aa_1, aa_2, \ldots, aa_n) + (ba_1, ba_2, \ldots, ba_n)\\ &= a(a_1, a_2, \ldots, a_n) + b(a_1, a_2, \ldots, a_n)\\ &= a\alpha + b\alpha\end{aligned}$$

(iii) If $a, b \in F$ and $\alpha = (a_1, a_2, \ldots, a_n) \in V$ then

$$\begin{aligned}(ab)\alpha &= (ab)(a_1, a_2, \ldots, a_n)\\ &= [(ab)a_1, (ab)a_2, \ldots, (ab)a_n]\\ &= [a(ba_1), a(ba_2), \ldots, a(ba_n)]\\ &= a(ba_1, ba_2, \ldots, ba_n)\\ &= a[b(a_1, a_2, \ldots, a_n)]\\ &= a(b\alpha)\end{aligned}$$

(iv) If 1 is the unity element of F and $\alpha = (a_1, a_2, \ldots, a_n) \in V$ then

$$1 \cdot \alpha = 1 \cdot (a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n) = \alpha$$

Hence V is a vector space over the field F. The vector space of all ordered n-tuples over F will be denoted by $V_n(F)$.

Example 2. *Prove that the set of all vectors in a plane over the field of real member is a vector space.*

Proof: Let V be the set of all vectors in a plane and R be the field of real numbers.

Then we observe that

1. **(V, +) is abelian group:**
   - **(i) Closure property :** Let $\alpha, \beta \in V \Rightarrow \alpha + \beta \in V$
   - **(ii) Commutative property :**Let $\alpha, \beta \in V$ then

     $$\alpha + \beta = \beta + \alpha, \forall \alpha, \beta \in V$$
   - **(iii) Associative property :** $(\alpha + \beta) + \Upsilon = \alpha + (\beta + \Upsilon), \forall \alpha, \beta, \Upsilon \in V$
   - **(iv) Existance of Identity :** Zero vector O in V such that

     $$\alpha + 0 = \alpha, \forall \alpha \in V$$
   - **(v) Existance of inverse :** If $\alpha \in V$, then the vector $-\alpha \in V$ such that

     $$\alpha + (-\alpha) = 0$$
2. If $\alpha \in V$ and $m \in R$ (m is any scalar). Then the scalar multiplication

   $$m\,\alpha \in V$$
3. Scalar multiplication and addition of vectors satisfy the following properties:
   - (i) $m(\alpha + \beta) = m\alpha + m\beta, \forall m \in R, \forall \alpha, \beta \in V$
   - (ii) $(m+n)\alpha = m\alpha + n\alpha\ \forall m, n \in R, \forall \alpha \in V$
   - (iii) $(mn)\alpha = m(n\alpha)\ \forall m, n \in R, \forall \alpha \in V$
   - (iv) $1 \cdot \alpha = \alpha, \forall \alpha \in V$ and 1 is the unit element of field R.

   Hence V is a vector space over the field R.

Example 3. *Let R be the field of real numbers and let R_n be the set of all polynomials over the field R. Prove that R_n is a vector space over the field R. Where R_n is of degree at most n.*

Solution. Here R_n is the set of polynomials of degree at most n over the field R. The set R_n is also includes the zero polynomial.

So, $R_n = \{f(x) : f(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n,$

Where $a_0, a_1, a_2, \ldots a_n \in R\}$

If $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

$g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$

$r(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$

Then, $f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n \in R_n$

Because it is also a polynomial of degree at most n over the field R.

Thus R_n is close for addition of polynomials.

∵ Addition of polynomials is commutative as well as associative. The zero polynomial 0 is a member of R_n and is identity for addition of polynomials.

Again if $f(x) = a_0 + a_1x + \ldots\ldots + a_nx^n \in R_n$

then $-f(x) = -a_0 - a_1x - a_2x^2 \ldots\ldots - a_nx^n \in R_n$

because it is also a polynomials of degree at most n over the field R.

We have $-f(x) + f(x) =$ zero polynomial.

The polynomial $-f(x)$ is the inverse of $f(x)$ for addition of polynomials.

Hence R_n is an addition group for addition of polynomials.

Now we define scalar multiplication $c\,f(x)$ by the relation.

$$cf(x) = ca_0 + (ca_1)x + (ca_2)x^2 + \ldots\ldots + (ca_n)x^n$$

Clearly $cf(x) \in R_n$ because it is also a polynomial of degree at most n over the field R. Then R_n is closed for scalar multiplication.

Now if $k_1, k_2 \in R$ and $f(x), g(x) \in R_n$ we have

$$k_1[(f(x) + g(x)] = k_1 f(x) + k_2 g(x)$$

$$(k_1 + k_2) f(x) = k_1 f(x) + k_2 f(x)$$

and $(k_1k_2) f(x) = k_1 [k_2 f(x)]$ can be proved easily.

Also $1 \cdot f(x) = f(x), f(x) \in R_n$

Hence R_n is a vector space over the field R.

General properties of vector spaces: Let $V(f)$ be a vector space over field F and $\overline{0}$ be the zero vector of V· then,

(i) $a.\overline{0} = \overline{0}$, $\forall\, a \in F$

(ii) $a\alpha = \overline{0}$, $\forall\, a \in V$

(iii) $a(-\alpha) = -(a\alpha)$, $\forall\, a \in F, \alpha \in V$

(iv) $(-a)\,\alpha = -(a\alpha)$, $\forall\, a \in F, \alpha \in V$

(v) $a(\alpha - \beta) = a\alpha - a\beta$ $\forall\, a \in F, \alpha, \beta \in V$

(vi) $a = \overline{0} \Rightarrow a = 0$ or $\alpha = \overline{0}$

Proof :

(i) We have, $a\bar{0} = a(\bar{0} + \bar{0})$

$= a(\bar{0}) + a(\bar{0})$

$\therefore$ $\bar{0} + a\bar{0} = a\bar{0} + a\bar{0}$

$\because$ V is an abelian group with respect to addition therefore by right cancellalion law in V, we get $\bar{0} = a \cdot \bar{0}$

(ii) $0\alpha = (0 + 0)\alpha$ [$\because 0 + 0 = 0 \in F$, by distributive law]

$= 0\alpha + 0\alpha$

$\bar{0} + 0\alpha = 0\alpha + 0\alpha$

By right cancellation law in V, we get

$\bar{0} = 0\alpha$

(iii) $a[\alpha + (-\alpha)] = a\alpha + a(-\alpha)$

$a \cdot \bar{0} = a\alpha + a(-\alpha)$

$\Rightarrow$ $\bar{0} = a\alpha + a(-\alpha)$

$\Rightarrow$ $a(-\alpha) = -(a\alpha)$

(iv) Now, $[a + (-a)]\alpha = a\alpha + (-a)\alpha$

$\Rightarrow$ $0\alpha = a\alpha + (-a)\alpha$

$\Rightarrow$ $\bar{0} = a\alpha + (-a)\alpha$

$(-a)\alpha$ is the additive inverse of $a\alpha$

$\Rightarrow$ $(-a)\alpha = -(a\alpha)$

(v) We have, $a(\alpha - \beta) = a[\alpha + (-\beta)]$

$= a\alpha + a(-\beta)$

$= a\alpha + [-(a\beta)]$ [$\because a(-\beta) = -(a\beta)$]

$= a\alpha - a\beta$

(vi) Let $a\alpha = \bar{0}$ then we have to prove that either $a = 0$ or $\alpha = \bar{0}$.

Let $a\alpha = \bar{0}$ and $a \neq 0 \in F$, so $\hat{a}$ exists

Then $\hat{a}(a\alpha) = \hat{a} \cdot \bar{0}$

$\Rightarrow$ $(\hat{a}a)\alpha = \bar{0}$

$\Rightarrow$ $1 \cdot \alpha = \bar{0}$

$\Rightarrow \quad \alpha = \overline{0}$

So when $a \neq 0$ then $\alpha = \overline{0}$

again let $a\alpha = \overline{0}$ and we have to prove that a = 0. Suppose $a \neq 0$ then $\hat{a}$ exists.

Now, $\quad a\alpha = \overline{0}$

$\hat{a}(a\alpha) = \hat{a} \cdot \overline{0}$

$(\hat{a}a)\alpha = \overline{0}$

$\Rightarrow \quad 1\cdot\alpha = \overline{0}$

$\Rightarrow \quad \alpha = \overline{0}$

Which is a contradictions that α must be a zero vector. Therefore $a = 0$

Hence $a\alpha = \overline{0}$ then either $a = 0$ or $\alpha = \overline{0}$.

Vector Subspace: Let *V* be a vector space over the field *F* and *W* be a subset of *V*. then *W* is said to be vector subspace of *V* if *W* is also a vector space with scalar multiplication and vector addition over the field *F* as *V*.

Some basic theorems of vector subspaces

***Theorem* 1:** The necessary and sufficient condition for a non empty subset *W* of a vector space *V (f)* to be subspace of *V* is that *W* is closed under vector addition and scalar multiplication.

Proof: Condition is necessary. Let *V* be a vector space and *W* be subspace of *V* over the same field *F*. Since W is vector sub space of *V*, so it is also a vector space under vector addition and scalar multiplications so it is closed. Hence condition is necessary.

The condition is sufficient: Let *V* be a vector space over field *F* and *W* be a non empty subset of *V*, such that *W* is closed under vector addition and scalar multiplication then we have to prove that *W* is subspace of *V*. For this we will prove that it is a vector space over field itself.

Let α∈W, if 1 is the unit element of *F* then -1∈ *F*. Now *W* is closed under scalar multiplication. Therefore,

$$(-1) \in F,\ \alpha \in W \Rightarrow (-1)\alpha \in W \Rightarrow -(1\alpha) \in W \Rightarrow -\alpha \in W$$

Thus, the additive inverse of each element of *W* is also in *W*. Now, *W* is closed under vector addition. Therefore

$$\alpha \in W \Rightarrow -\alpha \in W \Rightarrow \alpha + (-\alpha) \in W \Rightarrow \overline{0} \in W$$

Where $\overline{0}$ is the zero vector of *V*. Hence the zero vector of *V* is also the zero vector of *W*. Since $W \subseteq V$ therefore vector addition will be commutative as well as associative in W. Hence *W* is an abelian group with respect to vector addition. Also it is given that *W* is closed under scalar multiplication. The remaining properties of a vector space will hold in W. since they hold in *V* of which *W* is a subset. Hence *W* is itself a vector space with respect to vector addition and scalar multiplication as in *V,* so *W* is subspace of *V*. Hence condition in sufficient.

Theorem 2: The necessary and sufficient condition for a non empty subset *W* of a vector space *V(f)* to be a subspace of *V* is

$$a, b \in F, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$$

Proof: ***The condition is necessary:*** let *V* be a vector space over field *F* and *W* is subspace of *V*; then by the definition of subspace, *W* is a vector space over field *F* itself as *V*.

So, $\forall a \in F, \alpha \in W \Rightarrow a\,\alpha \in W$

$\forall b \in F, \beta \in W \Rightarrow b\,\beta \in W$

$\forall a\alpha \in W, b\beta \in W$, by vector addition in *W*, $a\alpha + b\beta \in W$

So condition is necessary.

The sufficient condition: Suppose *V* is a vector space over field *F* and *W* is nonempty subset of *V* such that $\forall a, b \in F, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$ then we have to show that *W* is subspace of *V,* for this we will show that *W* is a vector space itself as *V*.

$\because$ $a\alpha + b\beta \in W$

Put $a = b = 1 \in F$

$\Rightarrow$ $1 \cdot \alpha + 1 \cdot \beta \in W$

$\Rightarrow$ $\alpha + \beta \in W, \forall\, \alpha, \beta \in W$

So *W* is closed under vector addition.

Now taking *a* = 0, *b* = 0, we see that if

$\alpha \in W$ then

$0\alpha + 0\beta \in W$

$\overline{0} \in W$

Thus the zero vector of *V* belongs to *W*. It will also be the zero vector of *W*.

Now again $1 \in F, -1 \in F$

Taking $a=-1, b=0$

We get $-1\cdot\alpha+0\overline{0}\in W$

$\Rightarrow$ $-\alpha\in W$

Thus the additive inverse of each elements of *W* is also in *W*.

Now taking $\beta=\overline{0}$, we see that if $a,b\in F$ and $\alpha\in W$, then $a\alpha+b\overline{0}\in W$ *i.e.*, $a\alpha\in W$.

Thus *W* is closed under scalar multiplication. The remaining properties of a vector space will hold in *W* since they hold in *V* of which *W* is a subset.

Hence *W* is vector space itself. So by the definition *W* will be sub space of *V*.

***Theorem* 3.** The necessary and sufficient conditions for a non-empty subset *W* of a vector space *V (F)* to be a subspace of *V* are

(i) $\alpha\in W, \beta\in W\Rightarrow\alpha-\beta\in W$

(ii) $a\in F, \alpha\in W\Rightarrow a\alpha\in W$

Proof: As theorem 2.

***Theorem* 4.** *V* be a vector space and *W* is non empty subset of *V* then *W* will be sub space of *V* if and only if $\forall\alpha,\beta\in W,\ a\in F\Rightarrow a\alpha+\beta\in W$.

Proof: As theorem 2.

Examples on Vector Sub Spaces

***Example* 1.** *The set W of ordered trails* $(k_1,k_2,0)$ *where* $k_1,k_2\in F$ *is a subspace of* V_3 *(F).*

Solution. Let $\alpha=(k_1,k_2,0)$, and $\beta=(l_1,l_2,0)$ be any two element of *W*. Where $k_1,k_2,l_1,l_2\in F$. If *a*, *b* be any two elements of *F*, we have

$$a\alpha+b\beta=a(k_1,k_2,0)+b(l_1,l_2,0)$$

$$=(ak_1,ak_2,0)+(bl_1,bl_2,0)$$

$$=(ak_1+bl_1,ak_2+bl_2,0)$$

$\because$ $ak_1+bl_1, ak_2+bl_2\in F$ so

$a\alpha+b\beta\in W$

Hence *W* is a subspace of $V_3(F)$.

Example 2. *Prove that the set of all solution (l ,m, n) of the equation l+m+2n=0 is a subspace of the vector space V_3 (R)*

Solution. Let $W = \{(l, m, n) : l, m, n \in R \; and \; l + m + 2n = 0\}$

To prove that *W* is a subspace of V_3 (*R*) or R^3.

Let $\alpha = (l_1, m_1, n_1) \; and \; \beta = (l_2, m_2, n_2)$ be any two elements of *W*. Then

$$l_1 + m_1 + 2n_1 = 0$$

$$l_2 + m_2 + 2n_2 = 0$$

If a, b be any two elements of *R*, we have

$$a\alpha + b\beta = a(l_1, m_1, n_1) + b(l_2, m_2, n_2)$$

$$= (al_1, am_1, an_1) + (bl_2, bm_2, bn_2)$$

$$= (al_1 + bl_2, am_1 + bm_2, an_1 + bn_2)$$

Now $(al_1 + bl_2) + (am_1 + bm_2) + 2(an_1 + bn_2)$

$$= a(l_1 + m_1 + 2n_1) + b(l_2 + m_2 + 2n_2)$$

$$= a \cdot 0 + b \cdot 0 = 0$$

So $a\alpha + b\beta = (al_1 + bl_2, am_1 + bm_2, an_1 + bn_2) \in W$

Thus, $\alpha, \beta \in W and \; a, b \in R \Rightarrow a\alpha + b\beta \in W.$

Hence *W* is a subspace of $V_3(R)$.

***Example* 3.** If *V* is a vector space of all real valued continuous functions over the field of real numbers *R*, then show that the set *W* of solutions of the differential equation.

$\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$ is a subspace of *V*.

Solution. We have $W = \left\{ y : \dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0 \right\}$

It is clear that *y* = 0 satisfies the given differential equation and as such it belongs to *W* and thus $W \neq \Phi$.

Now let $y_1, y_2 \in W$ then

$$\frac{d^2 y_1}{dx^2} - 7\frac{dy_1}{dx} + 12 y_1 = 0 \qquad \dots \text{(i)}$$

$$\frac{d^2 y_2}{dx^2} - 7\frac{dy_2}{dx} + 12 y_2 = 0 \qquad \dots \text{(ii)}$$

Let a, b$\in R$. If W is to be subspace then we should show that $ay_1 + by_2$ also belongs to W *i.e.* It is a solution of the given differential equation. We have

$$\frac{d^2}{dx^2}(ay_1 + by_2) - 7\frac{d}{dx}(ay_1 + by_2) + 12(ay_1 + by_2)$$

$$= a \cdot \left(\frac{d^2 y_1}{dx^2} - 7\frac{dy_1}{dx} + 12 y_1\right) + b\left(\frac{d^2 y_2}{dx^2} - 7\frac{dy_2}{dx} + 12 y_2\right)$$

$$= a \cdot 0 + b \cdot 0$$

Thus a y_1 + by_2 is a solution of the given differential equation and so it belongs to W. Hence W is a subspace of V.

Algebra of subspaces

Theorem 1. The intersection of any two subspaces W_1 and W_2 of a vector space *V (f)* is also a subspace of *V (f)*.

Proof : Let V be a *vector space* over field F and W_1, W_2 are two subspaces of V. It is clear that $\overline{0} \in W_1 \text{ and } \overline{0} \in W_2 \text{ so } W_1 \cap W_2 \neq \Phi$

Let, $\alpha, \beta \in W_1 \cap W_2 \text{ and } a, b \in F$

$$\alpha \in W_1 \cap W_2 \Rightarrow \alpha \in W_1 \text{ and } \alpha \in W_2$$

$$\beta \in W_1 \cap W_2 \Rightarrow \beta \in W_1 \text{ and } \beta \in W_2$$

∵ W_1 is subspace of V so

$$\forall a, b \in F, \alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$$

$$\forall a, b \in F, \alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$$

So, $a\alpha + b\beta \in W_1 \cap W_2$

So, $W_1 \cap W_2$ is subspace of V.

Theorem 2. The union of two subspaces is a subspace if and only if one is contained in the other .

Proof: Suppose W_1 and W_2 are two subspaces of V.

Condition is necessary: Let $W_1 \subseteq W_2$ *or* $W_2 \subseteq W_1$, then we will prove that $W_1 \cup W_2$ will be subspace of V.

If $W_1 \subseteq W_2 \Rightarrow W_1 \cup W_2 = W_2$ *and if* $W_2 \subseteq W_1 \Rightarrow W_1 \cup W_2 = W_1$.

But W_1 and W_2 both are subspace of V, so $W_1 \cup W_2$ will be subspace of V.

Condition is sufficient: Let W_1 and W_2 be two subspaces of V such that $W_1 \cup W_2$ be also subspace of V, we have to show that $W_1 \subseteq W_2$ *or* $W_2 \subseteq W_1$. Let us assume that W_1 is not a sub set of W_2 and W_2 is also not a subset of W_1.

$\therefore$ $W_1 \subseteq W_2 \Rightarrow \exists \alpha \in W_1$ such that $\alpha \notin W_2$

and $W_2 \subseteq W_1 \Rightarrow \exists \beta \in W_2$ such that $\beta \notin W_1$

But, $\alpha \in W_1 \cup W_2$

and $\beta \in W_1 \cup W_2$

But $W_1 \cup W_2$ is subspace of V so

$$\alpha + \beta \in W_1 \cup W_2$$

$\Rightarrow$ $\alpha + \beta \in W_1$ *or* $\alpha + \beta \in W_2$

If $\alpha + \beta \in W_1$ *and* $\alpha \in W_1, -\alpha \in W_1$

$\Rightarrow$ $(\alpha + \beta) - \alpha \in W_1 \Rightarrow \beta \in W_1$

Which is a contradiction, again if

$\Rightarrow$ $\alpha + \beta \in W_2$ *and* $-\beta \in W_2$

$\Rightarrow$ $(\alpha + \beta) + (-\beta) \in W_2$

$\Rightarrow$ $\alpha \in W_2$

Again we get a contradiction. Hence either $W_1 \subseteq W_2$ *or* $W_2 \subseteq W_1$.

Theorem 3. Intersection of any family of subspaces of a vector space is a subspace.

Proof: As above

Linear combination : Let *V* (*f*) be a vector space if $\alpha_1, \alpha_2, ...\alpha_n \in V$ then any vector $\alpha \in V$.

$\alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$ where $a_i \in F$ is called a linear combination of the vectors $\alpha_1, \alpha_2 ...,....\alpha_n$.

Linear span : Let *V* (*f*) be a vector space and *S* be any non empty subset of *V*. Then the linear span of *S* is the set of all linear combinations of finite sets of elements of *S* and is denoted by *L*(*S*). Thus we have

$$L(S) = \{a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n, \alpha_i \in V, a_i \in F\}$$

Linear dependence and linear independence. Let *V* be a vector space over field *F*. A finite set $S = \{\alpha_1, \alpha_2, ...\alpha_n\}$ is said to be linearly dependent if

$$a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \overline{0}$$

Where $\alpha_i \in V, a_i \in F$, and all a_i s may not zero. There will be minimum one $a_i \neq 0$.

A finite set $S = \{\alpha_1, \alpha_2, ...\alpha_n\}$ is said to be linearly independent if $a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \overline{0}$

Where α_i's ∈ *V* and a_i's $\in F$ and all a_i's = 0

Any infinite setof vectors of *V* is said to be linearly independent if its every finite subset is linearly independent, otherwise it is linearly dependent.

Example 4. Show that the vector *(1, 2, 0), (0, 3, 1), (-1, 0, 1)* forms linearly independent set over field *R*.

Solutions. Let,

$$S = \{\alpha_1, \alpha_2, \alpha_3\}$$

Where,

$$\alpha_1 = (1,2,0), \alpha_2 = (0,3,1), \alpha_3 = (-1,0,1)$$

Let $a_1, a_2, a_3 \in F$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \overline{0}$$

Then *S* will be linearly independent if all $a_1 = a_2 = a_3 = 0$

Now

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \overline{0}$$

$$a_1(1,2,0) + a_2(0,3,1) + a_3(-1,0,1) = (0,0,0)$$

$$(a_1, 2a_1, 0) + (0, 3a_2, a_2) + (-a_3, 0, a_3) = (0,0,0)$$

$$(a_1 - a_3, 2a_1 + 3a_2, a_2 + a_3) = (0,0,0)$$

$$a_1 + 0a_2 - a_3 = 0$$

$$2a_1 + 3a_2 + 0a_3 = 0$$

$$0a_1 + a_2 + a_3 = 0$$

Coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}, |A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1[3] - 0 - 1[2]$$

$$|A| = 1 \neq 0$$

∴ Rank A = 3, hence there will be only one solution $a_1 = a_2 = a_3 = 0$

Hence $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent.

Example 5. Show that $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly dependent over field ***R***. Where $\alpha_1 = (1,3,2), \alpha_2 = (1,-7,-8), \alpha_3 = (2,1,-1)$.

Proof: Let $a_1, a_2, a_3 \in R$

Now, $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \overline{0}$

$$\Rightarrow \quad a_1(1,3,2) + a_2(1,-7,-8) + a_3(2,1,-1) = (0,0,0)$$

$$\Rightarrow \quad (a_1 + a_2 + 2a_3, 3a_1 - 7a_2 + a_3, 2a_1 - 8a_2 - a_3) = (0,0,0)$$

$$a_1 + a_2 + 2a_3 = 0$$

$$3a_1 - 7a_2 + a_3 = 0$$

$$2a_1 - 8a_2 - a_3 = 0$$

Coefficient matrix

$$A=\begin{bmatrix}1 & 1 & 2\\ 3 & -7 & 1\\ 2 & -8 & -1\end{bmatrix}$$

$$|A|=\begin{vmatrix}1 & 1 & 2\\ 3 & -7 & 1\\ 2 & -8 & -1\end{vmatrix}$$

$$=1[7+8]-1[-3-2]+2[-24+14]$$

$$=15+5-20=0$$

$$|A|=0$$

So rank of A<3

Rank of A<number of unknowns

So there is minimum one $a_i \neq 0$

So $S=\{\alpha_1,\alpha_2,\alpha_3\}$ is linearly dependent.

Example 6. If α,β,γ are linearly independent vectors of *V* (*f*) where *F* is any sub field of complex numbers than prove that $\alpha+\beta,\beta+\gamma,\gamma+\alpha$ are also linearly independent.

Solution. Let a_1,a_2,a_3 be scalar then

$$a_1(\alpha+\beta)+a_2(\beta+\gamma)+a_3(\gamma+\alpha)=\overline{0}$$

$$(a_1+a_3)\alpha+(a_1+a_2)\beta+(a_2+a_3)\gamma=\overline{0} \quad \ldots(i)$$

But α,β,γ are linearly independent. Therefore (i) implies

$$a_1+0a_2+a_3=0$$

$$a_1+a_2+0a_3=0$$

$$0a_1+a_2+a_3=0$$

The coefficient matrix A of these equations is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$|A| = 1[0] + 0 + 1[1-0] = 1 \neq 0$$

Rank A = 3 = number of unknowns

There is only one solution

$$a_1 = a_2 = a_3 = 0$$

So $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ are also linearly independent.

Basic of a vector space : A subset $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ of a vector space $V(f)$ is said to be a basis of $V(f)$ if

(i) S consists of linearly independent vectors *i.e.,*

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \overline{0}$$

all a_i's are zero, $\forall a_i \in F, \alpha_i \in V$

(ii) $L(S) = V(f)$ *i.e.,* every element of V can be written as linear combination of element of S.

Example 7. Show $S = \{(1,2,1),(2,1,0),(1,-1,2)\}$ forms a basis of R^3.

Proof : Since dim $R^3 = 3$

So $L(S) = V(R^3)$

Now we only to prove that S is linearly independent

Let $a_1, a_2, a_3 \in F$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \overline{0}$$

We will prove that $a_1 = a_2 = a_3 = 0$

Now $a_1(1,2,1) + a_2(2,1,0) + a_3(1,-1,2) = (0,0,0)$

$$(a_1 + 2a_2 + a_3, 2a_1 + a_2 - a_3, a_1 + 0a_2 + 2a_3) = (0,0,0)$$

$$a_1 + 2a_2 + a_3 = 0$$

$$2a_1 + a_2 - a_3 = 0$$

$$a_1 + 0a_2 + 2a_3 = 0$$

Coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix}$$

$= 1\,[2 - 0] - 2\,[4 + 1] + 1\,[0 - 1]$

$= 2 - 10 - 1$

$\neq 0$

So rank of $A = 3$ = no of unknown

So there is only one solution $a_1 = a_2 = a_3 = 0$

So $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent

So S forms the basis of R^3.

Example 8. Select a basis of R^3 (R) from the set $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where

$$\alpha_1 = (1, -3, 2), \alpha_2 = (2, 4, 1), \alpha_3 = (3, 1, 3), \alpha_4 = (1, 1, 1)$$

Solution. If any three vectors in S are linearly independent, then they will form a basis of the vector space R^3 (R).

First we take $S_1 = \{\alpha_1, \alpha_2, \alpha_3\}$

For this we take $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \overline{0}$

i.e., $a_1(1, -3, 2) + a_2(2, 4, 1) + a_3(3, 1, 3) = (0, 0, 0)$

$$(a_1 + 2a_2 + 3a_3, -3a_1 + 4a_2 + a_3, 2a_1 + a_2 + 3a_3) = (0, 0, 0)$$

$$a_1 + 2a_2 + 3a_3 = 0$$

$$-3a_1 + 4a_2 + a_3 = 0$$

$$2a_1 + a_2 + 3a_3 = 0$$

Coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$|A| = 0$$

So rank of A<3

i.e., rank of A < no. of unknowns

So $S_1 = \{\alpha_1, \alpha_2, \alpha_3\}$ are linearly dependent

Now we take $S_2 = \{\alpha_1, \alpha_2, \alpha_4\}$

Then we get

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ -3 & 4 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

So rank of A = No. of unknown

So $[\alpha_1, \alpha_2, \alpha_4]$ is linearly independent

So $S_2 = \{\alpha_1, \alpha_2, \alpha_4\}$ forms the basis of R^3 (R).

Linear Transformation: Let *U*(*f*) and *V* (*f*) be two vector space over the same field $F.T : U \to V$ is said to be linear transformation if

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \qquad \ldots(i)$$

$\forall \alpha, \beta$ in *U* and *a*, *b* in *F*

in another way the properly (i) can be defined in two ways

(i) $T(\alpha + \beta) = T(\alpha) + T(\beta)$ and

(ii) $T(a\alpha) = aT(\alpha), \forall a \in F$, $\alpha, \beta \in U$

Example 9. The function $T : V_3(R) \to V_2(R)$ defined by $T(a,b,c) = (a,b), \forall a,b,c \in R,$ is a linear transformation from V_3 (R) into V_2 (R).

Proof: If $T : V_3(R) \to V_2(R)$ will be linear transformation then

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$$

Let $\alpha = (a_1, b_1, c_1) \in V_3(R)$

$\beta = (a_2, b_2, c_2) \in V_3(R)$ and $a, b \in R$

Now $a\alpha + b\beta = a(a_1, b_1, c_1) + b(a_2, b_2, c_2)$

$$= \left(aa_1, ab_1, ac_1\right) + (ba_2, bb_2, bc_2)$$

$$= \left(aa_1 + ba_1, ab_2 + bb_2, ac_1 + bc_2\right)$$

L.H.S. $= T\left(a\alpha + b\beta\right)$

$$= T\left(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2\right)$$

$= (aa_1 + ba_2, ab_1 + bb_2)$, by def. of T

$$= \left(aa_1, ab_1\right) + \left(ba_2, bb_2\right)$$

$$= a(a_1, b_1) + b(a_2, b_2)$$

$$= aT(a_1, b_1, c_1) + bT(a_2, b_2, c_2)$$

$$= aT(\alpha) + bT(\beta) = R.H.S.$$

So T is linear transformation from $V_3(R) \to V_2(R)$

Range and Null space of a linear transformation

Let U and V be two vector spaces over the same field F and let T be a linear transformation from U into V.

Then the range of T is written as $R(T)$ and it is the set of all vectors β is V such that $\beta = T(\alpha)$ for some $\alpha \in U$.

Range $T = \left\{T\left(\alpha\right) = \beta : \alpha \in U, \beta \in V\right\}$

Null space of a linear transformation

Let U and V be two vector spaces over same field F and let T be a linear transformation from U into V. Then the null space of T is written as $N(T)$ and it is the set of all vectors α in U such that $T\left(\alpha\right) = \overline{0}$ for some $\alpha \in U$.

That is $N(T) = \left\{\alpha \in U : T(\alpha) = \overline{0} \in V\right\}$

It is to be noted that if we take *T* as vector space homomorphism of *U* into *V*, then the null space of *T* is also called the karnel of *T*.

Rank and nullity of a linear transformation

Let *U* and *V* be two vector spaces over the same field *F* and *T* be a linear transformation from *U* to *V*, with *U* as finite dimensional.

The rank of *T* is denoted by $\rho(T)$ and it is the dimension of the range of *T i.e.*,

$$\rho(T) = \dim R(T)$$

The nullity of *T* is denoted by $v(T)$ and it is the dimension of null space of *T i.e.*, $v(T) = \dim N(T)$

Note: Let $T: U(F) \to V(F)$ be a linear transformation from *U* into *V*. Suppose that *U* is finite dimensional. Then rank *T* + nullity *T* = dim *U*.

Example 10. Show that the mapping $T: V_2(R) \to V_3(R)$ defined as $T(a,b) = (a+b, a-b, b)$ is a linear transformation from $V_2(R) \to V_3(R)$. Find the range, rank, null space and nullity of *T*.

Solution: Let $\alpha = (a_1, b_1)$, $\beta = (a_2, b_2)$ be arbitrary elements of $V_2(R)$. Then $T: V_2(R) \to V_3(R)$ will be a linear transformation if

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta), \forall a, b \in R$$

$$\alpha, \beta \in V_2(R), a, b \in R \text{ then } a\alpha + b\beta \in V_2(R)$$

Now, $T(a\alpha + b\beta) = T\ [a(a_1, b_1) + b(a_2, b_2)]$

$$= T\ [(aa_1, ab_1) + (ba_2, bb_2)]$$

$$= T\left[(aa_1 + ba_2, ab_1 + bb_2)\right]$$

$$= \left[(aa_1 + ba_2) + (ab_1 + bb_2), (aa_1 + ba_2) - (ab_1 + bb_2), (ab_1 + bb_2)\right]$$

$$= \left[a(a_1 + b_1) + b(a_2 + b_2), a(a_1 - b_1) + b(a_2 - b_2), (ab_1 + bb_2)\right]$$

$$= a(a_1 + b_1, a_1 - b_1, b_1) + b(a_2 + b_2, a_2 - b_2, b_2)$$

$$= aT(a_1, b_1) + bT(a_2, b_2)$$

$$= aT(\alpha) + bT(\beta)$$

So $\quad T\left(a\alpha + b\beta\right) = aT(\alpha) + bT(\beta)$

So T is a linear transformation from $V_2(R)$ into $V_3(R)$.

Now {(1,0),(0,1)} is a basis for $V_2(R)$

We have, $T(1, 0) = (1 + 0, 1 – 0, 0) = (1, 1, 0)$

$T(0, 1) = (0 + 1, 0 – 1, 0) = (1, - 1, 1)$

The vectors $T(1, 0)$, $T(0, 1)$ span the range of T.

Thus the range of T is sub space of $V_3(R)$ spanned by the vectors (1, 1, 0) and (1, -1, 1).

Now the vectors $(1,1,0),(1,-1,1) \in V_3(R)$ are linearly independent if $x, y \in R,$

Then

$$x(1,1,0) + y(1,-1,1) = (0,0,0)$$

$$\Rightarrow \quad (x + y, x - y, y) = (0,0,0)$$

$$\Rightarrow \quad x + y = 0, x - y = 0, y = 0$$

$$\Rightarrow \quad x = 0, y = 0$$

The vectors (1, 1, 0), (1, -1, 1) form a basis for range of T.

Hence rank T = dim of range of T = 2

Nullity of T = dim of $V_2(R)$ – rank of T = 2 – 2 = 0

Null space of T must be the zero subspace of $V_2(R)$

Otherwise $(a,b) \in$ null space of T

$$\Rightarrow \quad T(a,b) = (0,0,0)$$

$$(a + b, a - b, b) = (0,0,0)$$

$$a + b = 0$$

$$a – b = 0$$

$$b = 0$$

$$\Rightarrow \quad a = 0, b = 0$$

∴ (0,0) is the only element of $V_2(R)$ which belong to null space of T.

∴ Null space of T is the zero subspace of V_2 (R).

Representation of transformation by matrices

Let U be an n-dimensional vector space over the field F and let V be an m-dimensional vector space over the field F.

We take two ordered basis

$$\beta = \{\alpha_1, \alpha_2, ...\alpha_n\} \text{ and } \beta' = \{\beta_1, \beta_2, ...\beta_m\}$$

for U and V respectively

Let $T : U \to V$ be a linear operator: since T is completely determined by its action on the vectors α_j belonging to a basis for U. Each of the n vectors $T(\alpha_j)$ is uniquely expressible as a linear combination of $\beta_1, \beta_2, ...\beta_m$. For j = 1, 2,……n.

Then, $$T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + ... + a_{mj}\beta_m = \sum_{i=1}^{m} a_{ij}\beta_i$$

The scalars $a_{1j}, a_{2j}, ...a_{mj}$ are the co-ordinates of $T(\alpha_j)$ in the ordered basis β' .

The m x n matrix whose j^{th} column (j = 1, 2, …….n) consists of these co-ordinates is called the matrix of the linear transformation T relative to the pair of ordered basis β and β\` . It is denoted by the symbol $[T : \beta : \beta']$ or simply by [T] if the basis is understood. Thus,

[T] = $[T : \beta : \beta']$ = matrix of T relative to ordered basis β and $\beta' = [a_{ij}]_{m \times n}$

and $$T(\alpha_j) = \sum_{i=1}^{m} a_{ij}\beta_i, \ \forall\ j = 1, 2, ...n$$

Example 11. Find the matrix of the linear transformation T on V_3 (R) defined as T (x, y, z) = (2y + z, x - 4y, 3x) with respect to the ordered basis β and also with respect to the ordered basis β\` where

(i) β = {(1, 0, 0), (0, 1, 0), (0, 0, 1)}

(ii) β\` = {(1, 1, 1), (1, 1, 0), (1, 0, 0)}

Solution.

(i) We have

T (1, 0, 0) = (0, 1, 3) = 0 (1, 0, 0) + 1 (0, 1, 0) + 3 (0, 0, 1)

T (0, 1, 0) = (2 – 4, 0) = 2(1, 0, 0) - 4 (0, 1, 0) + 0 (0, 0, 1)

and T (0, 0, 1) = (1, 0, 0) = 1 (1, 0, 0) + 0 (0, 1, 0) + 0 (0, 0, 1) so by def of matrix of *T*, with respect to β , we have

$$[T]_\beta = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

(ii) We have *T* (1, 1, 1) = (3, -3, 3)

We have to express (3, -3, 3) as a linear combination of vectors in β\`.

Let $(a,b,c) = x_1(1,1,1) + y_1(1,1,0) + z_1(1,0,0)$

$$= (x_1 + y_1 + z_1,\ x_1 + y_1, x_1)$$

$$x_1 + y_1 + z_1 = a,\ x_1 + y_1 = b, x_1 = c$$

So $x_1 = c, y_1 = b - c, z_1 = a - b$

For (3, -3, 3), putting *a* = 3, *b* = - 3, *c* = 3

$$x_1 = 3, y_1 = -6 \; and \; z_1 = 6 \qquad \ldots \text{(i)}$$

So, *T* (1, 1, 1) = (3, -3, 3) = 3 (1, 1, 1) – 6 (1, 1, 0) + 6 (1, 0, 0)

Also, *T* (1, 1, 0) = (2, -3, 3)

Putting *a* = 2, *b* = -3 and *c* = 3 in (i) we get

T(1, 1, 0) = (2, -3, 3) = 3 (1, 1, 1) – 6 (1, 1, 0) + 6 (1, 0 , 0)

Similarly, *T*(1, 0, 0) = (0, 1, 3)

So, *a* = 0, *b* = 1, *c* = 3

T (1, 0, 0) = (0, 1, 3) = 3 (1, 1, 1) – 2 (1, 1, 0) – 1 (1, 0, 0)

So, $$[T]_{\beta'} = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

Practice Problems

1. Suppose *R* be the field of real numbers. Which of the following are subspace of $V_3(R)$:

   (i) {(*a*, 2*b*, 3*c*) : *a*, *b*, *c* ∈ *R*}, (ii) {(*a*, *a*, *a*) : *a* ∈ R}

   (iii) {(*a*, *b*, *c*): *a* ,*b*, *c* are rational numbers}

   Ans. (i) and (ii)

2. In $V_3(R)$, where R is the field of real numbers, examines each of the following sets of vectors for linear independence/ dependence.

   (i) {(2, 1, 2), (8, 4, 8)} (ii) {(-1, 2, 1), (3, 0, -1), (-5, 4, 3)}

   (iii) {(2, 3, 5), (4, 9, 25)} (iv) {(1, 2, 1), (3, 1, 5), (, -4, 7)}

   Ans. (i) Dependent, (ii) Dependent (iii) Independent (iv) Dependent.

3. Show that the three vectors (1, 1, -1), (2, -3, 5) and (-2, 1, 4) of R^3 are linearly independent.
4. Determine if the set {(2, -1, 0), (3, 5, 1), (1, 1, 2)} is a basis of $V_3(R)$.
5. Show that the vectors $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1), \alpha_3 = (0, -3, 2)$ form a basis of $V_3(R)$. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.
6. Show that the set {(1, i, 0), (2i, 1, 1), (0, 1+i, 1-i)} is a basis for $V_3(c)$.
7. Let $T : V_3(R) \to V_3(R)$ defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$

   What is the matrix of T in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1,1,0), \alpha_2 = (-1,2,1), \alpha_3 = (2,1,1)$.

$$\textbf{Ans.}\quad T = \frac{1}{4}\begin{bmatrix} 17 & 35 & 22 \\ -3 & 15 & -6 \\ 2 & -14 & 0 \end{bmatrix}$$