Unit - 1

Linear Systems of Equations

- Fundamental concepts
- Elementary row (and column) transformations
- Rank of a matrix - Echelon form, Normal form
- Solution of linear systems - direct methods
- L-U decomposition
- L-U decomposition from Gauss elimination
- Solution of Tridiagonal systems
- Gauss-Siedel Iterative Method
- Application: Finding the Current in Electrical Circuit
- A summary
- Solved University Questions
- Objective type Questions

1.1 Introduction

The concept of matrices was introduced by James Joseph Sylvester and the theory was further developed by Hamilton, Cayley and others. The problems connected with linear transformations, systems of linear equations can be simplified by employing matrix ideas. Matrix theory is a powerful tool to handle many complicated problems that occur in Pure and Applied Mathematics, Statistics, Applied Sciences and Engineering branches such as Electrical, Electronic, Communication etc., and Computer Science and Information Technology. The application of the concepts of matrix are also useful in other branches such as Chemistry, Physics, Quantum Mechanics, Sociology, Economics and Education.

1.1.1 Matrices - Fundamental Concepts

A rectangular array (arrangement) of 'mn' real or complex numbers in 'm' rows (horizontal) and 'n' columns (vertical) is called an m × n matrix (read as m by n matrix), m and n being positive integers.

A matrix is, in general, denoted by capital letters A, B, C, ...

Example :

(i) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 4 & 5 \\ -2 & 1 & 0 & 7 \end{bmatrix}$ is a 3 × 4 matrix as it has 3 rows and 4 columns.

(ii) $B = \begin{bmatrix} 1 & -1 & 0 \\ 6 & 5 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ is a 3 × 3 matrix.

Note : (i) The numbers in a matrix are arranged within symbols like [ ] or ( ) or || ||

(ii) Order of a Matrix : If a matrix has m rows and n columns then its order is m × n (read as m by n).

1.1.2 Notation

The matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

is denoted in a simple form as

$$A = (a_{ij})_{m\times n},$$

which means that, (i) the matrix has m rows and n columns and (ii) the number in the i^{th} row and j^{th} column is a_{ij} and (iii) $i = 1, 2, ...m$, $j = 1, 2, ... n$.

a_{ij} are called the elements of the matrix.

TYPES OF MATRICES :

1.1.3 Rectangular Matrix

If the number of rows 'm' and the number of columns 'n' of a matrix are unequal, it is known as a rectangular matrix $(\mathbf{m} \neq \mathbf{n})$.

Example :

$$A = \begin{bmatrix} 7 & 6 & 3 & 1 \\ -1 & 2 & 0 & 5 \\ 1 & 3 & 4 & -2 \end{bmatrix}$$

1.1.4 Square Matrix

If the number of rows and columns are equal it is said to be a square matrix **(m = n)**.

Example :

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

are square matrices of order 3 and 2 respectively.

1.1.5 Row Matrix

A matrix which has only one row is called a row matrix **(m = 1)**

Example :

$$\begin{bmatrix} 1 & -1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \end{bmatrix}$$

1.1.6 Column Matrix

A matrix which has only one column is called a column matrix **(n = 1)**

Example :

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 4 \end{pmatrix}$$

1.1.7 Null (zero) Matrix

A matrix of order m × n in which each element is equal to 'zero' is called a null matrix and denoted by $0_{m \times n}$ **($a_{ij} = 0 \forall$ i, j)**

Example :

$$0_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ are null matrices.}$$

1.1.8 Principal Diagonal

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n. Then the elements a_{ii}, (i = 1, 2, ... n) are known as the elements of the 'principal diagonal' or leading diagonal of the matrix.

Example :

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a given square matrix, then the elements a_{11}, a_{22}, a_{33} are called the elements of the principal diagonal.

1.1.9 Unit Matrix or Identity Matrix

If $a_{ij} = 1$, if i = j

and $a_{ij} = 0$ if i ≠ j, then the matrix $(a_{ij})_{n \times n}$ is called a unit matrix of order 'n' and is denoted by I_n.

i.e., If each of the elements of the principal diagonal are equal to unity and all other elements are equal to zero, then such a square matrix is called a unit matrix.

Example :

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ldots\ldots I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

1.1.10 Diagonal Matrix

If $A = (a_{ij})_{n \times n}$ is a square matrix such that all its non-diagonal elements are zero, then A is called a diagonal matrix or quasi-scalar matrix.

Example :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

It is also written as Diag [1, –2, 3].

1.1.11 Scalar Matrix

In a diagonal matrix, if all the elements of the diagonal are equal, it is called a scalar matrix.

Example :

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

are examples of scalar matrices.

1.1.12 Triangular Matrices

(i) ***Upper Triangular Matrix :***

If $a_{ij} = 0 \ \forall \ i > j$ in a square matrix $A = (a_{ij})_{n \times n}$, then A is called an upper triangular matrix. i.e., In an upper triangular matrix all the elements below the principal diagonal are equal to zero.

Example :

$$A = \begin{bmatrix} 4 & -3 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$ is an upper triangular matrix.

(ii) ***Lower Triangular Matrix :***

If $a_{ij} = 0, \ \forall, \ i < j$ in a square matrix $A = [a_{ij}]_{n \times n}$, then A is called a lower triangular matrix. i.e., In a lower triangular matrix all the elements above the principal diagonal are equal to zero.

Example :

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 8 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$ is a lower triangular matrix.

1.1.13 Equality of Matrices

Let A and B be two matrices. Then A = B if and only if

(i) Both A and B are of the same order, and

(ii) Each element of A is equal to the corresponding element of B.

i.e., if $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$, are two matrices such that

$a_{ij} = b_{ij}$, $\forall$ i and j,

then we say that A and B are equal or A = B.

Example :

If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and

$B = \begin{bmatrix} 2 & -3 \\ 4 & 0 \end{bmatrix}$, then,

$A = B \Leftrightarrow p = 2, q = -3, r = 4, s = 0$

Further, for any two matrices A and B, the inequalities A > B or A < B are not defined.

1.1.14 Trace of a Square Matrix

The trace of a square matrix $A = (a_{ij})_{n \times n}$, denoted as trace A, is defined by the sum of its diagonal elements, i.e., Trace $A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \ldots.. + a_{nn}$.

Example :

If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & -6 \\ 5 & 0 & 7 \end{bmatrix}$,

Trace $A = a_{11} + a_{22} + a_{33} = 1 + 4 + 7 = 12$.

RELATED MATRICES :

1.1.15 Transpose of a Matrix

The matrix obtained by interchanging the rows and columns of a given matrix A is called the transpose of A and is denoted by A′ or A^T.

Example :

(i) If $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ -1 & 3 & 2 & 0 \\ 6 & 0 & 1 & 1 \end{bmatrix}_{3\times4}$ then,

$$A'(\text{or } A^T) = \begin{bmatrix} 2 & -1 & 6 \\ 3 & 3 & 0 \\ 4 & 2 & 1 \\ 5 & 0 & 1 \end{bmatrix}_{4\times3}$$

(ii) If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 6 \\ 4 & 1 & -3 \end{bmatrix}_{3\times 3}$, $A' = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & 1 \\ 2 & 6 & -3 \end{bmatrix}_{3\times 3}$

Thus (i) If $A = (a_{ij})_{m\times n}$, then $A' = (a_{ij})_{n\times m}$. i.e., if A is an $m \times n$ matrix then A′ is an $n \times m$ matrix.

1.1.16 Symmetric Matrix

If A is a square matrix such that $A' = A$ then A is called a symmetric matrix. i.e., If $A = (a_{ij})_{m\times n}$ is a symmetric matrix, then $a_{ij} = a_{ji}\ \forall\ i, j$.

Thus a symmetric matrix is one which is unchanged by interchange of rows and columns.

Example :

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}, B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ are symmetric.}$$

1.1.17 Skew-Symmetric Matrix

If $A = [a_{ij}]_{n\times n}$ is a square matrix such that $A = -A'$, then A is called a skew-symmetric matrix.

Thus $a_{ij} = -a_{ji}\ \forall$ i and j.

If $i = j$, $a_{ii} = -a_{ii}$

$\Rightarrow$ $a_{ii} = 0$

i.e., the diagonal elements of a skew-symmetric matrix are all equal to zero.

Example :

$$\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$$

are skew-symmetric matrices.

1.1.18(1) Complex Matrix

If the elements of a matrix are complex numbers, it is known as complex matrix.

Example :

$$A = \begin{bmatrix} x+iy & z+it \\ a+ib & c+id \end{bmatrix}$$

A complex matrix can be expressed as a sum of two real matrices. In the above example,

$$A = \begin{bmatrix} x & z \\ a & c \end{bmatrix} + i\begin{bmatrix} y & t \\ b & d \end{bmatrix} = M + iN$$

where $x, y, a, b, \ldots,$ are all real numbers.

1.1.18(2) Conjugate of a Matrix

Let $A = (a_{ij})_{m \times n}$ be any given matrix. Then $\overline{A} = \left(\overline{a}_{ij}\right)_{m\times n}$ is called the conjugate of A, where $\overline{a}_{ij}$ is the complex conjugate of a_{ij}.

i.e., If all the elements of any given matrix A are replaced by their complex conjugates, then the resulting matrix is called the conjugate of A and is denoted by $\overline{A}$.

Example :

If $A = \begin{bmatrix} 3+i & 1 & 1+i & i \\ 2 & 2i & 2+3i & -1 \\ 0 & -i & 7+6i & 4 \end{bmatrix}$ then, $\overline{A} = \begin{bmatrix} 3-i & 1 & 1-i & -i \\ 2 & -2i & 2-3i & -1 \\ 0 & i & 7-6i & 4 \end{bmatrix}$ is called the conjugate of A.

1.1.19 Transposed Conjugate of a Matrix

Let $A = (a_{ij})$ be any given matrix. Then the transpose of the conjugate of A is called the transposed conjugate of A and is denoted by A^* or A^θ

Thus if $A = (a_{ij})_{m \times n}$, then $A^* = (c_{ji})_{n \times m}$, where $c_{ji} = \overline{a}_{ij}$.

Example :

$$A = \begin{bmatrix} 3+i & 1 & 1+i & i \\ 2 & 2i & 2+3i & -1 \\ 0 & -i & 7+6i & 4 \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} 3-i & 1 & 1-i & -i \\ 2 & -2i & 2-3i & -1 \\ 0 & i & 7-6i & 4 \end{bmatrix}$$

and $$A^* = \left(\overline{A}\right)' = \begin{bmatrix} 3-i & 2 & 0 \\ 1 & -2i & i \\ 1-i & 2-3i & 7-6i \\ -i & -1 & 4 \end{bmatrix}.$$

It can be also seen that $A^* = \overline{A'}$.

1.1.20 Hermitian Matrix

If A is a square matrix such that $A^* = A$, then A is said to be a Hermitian matrix.(JNTU 2003)

Let $A = (a_{ij})_{n \times n}$ be the given Hermitian matrix.

Then $A^* = \left(\bar{a}_{ji}\right)_{n \times n}$.

Since A is Hermitian,

$$a_{ij} = \bar{a}_{ji}, \ \forall \ i, j$$

In particular, if i = j, $a_{ii} = \bar{a}_{ii}$

$\therefore$ a_{ii} is real $\forall$ i

Thus, in a Hermitian matrix, all the diagonal elements are real.

Example :

$$\begin{bmatrix} 2 & 3+4i & 1-3i \\ 3-4i & -4 & i \\ 1+3i & -i & 0 \end{bmatrix}$$ is a Hermitian matrix.

1.1.21 Skew-Hermitian Matrix (JNTU 2003)

If A is a square matrix such that $A^* = -A$, then A is called a skew-Hermitian matrix.

Let $A = (a_{ij})$ be skew-Hermitian

then, $a_{ij} = -\bar{a}_{ji}$, $\forall$ i and j.

If $i = j$, $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0 \Rightarrow a_{ii}$ is purely imaginary or $a_{ii} = 0$.

$\therefore$ In a skew-Hermitian matrix, the diagonal elements must be either purely imaginary or equal to zero.

Example :

$$\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1+i & 2-3i \\ -1+i & 4i & 4+i \\ -2-3i & -4+i & i \end{bmatrix}$$ are examples of skew-Hermitian matrices.

1.1.22 Addition and Substraction of Matrices

If A and B are two matrices of the same order, then their sum (or difference) is another matrix of the same order in which each element is the sum (or difference) of the corresponding elements of A and B.

Thus, if $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$ are two given matrices of order $m \times n$, then,

$C = (c_{ij})_{m \times n} = A + B$

where $c_{ij} = a_{ij} + b_{ij}$, $\forall$ i and j,

and $D = (d_{ij})_{m \times n} = A - B$

where $d_{ij} = a_{ij} - b_{ij}$ $\forall$ i and j.

Example :

Let $A = \begin{bmatrix} 4 & 7 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$

then, (i) $A + B = \begin{bmatrix} 4+1 & 7-3 \\ 3+2 & 5+0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 5 & 5 \end{bmatrix}$

and (ii) $A - B = \begin{bmatrix} 4-1 & 7+3 \\ 3-2 & 5-0 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 1 & 5 \end{bmatrix}$

Note : (i) Matrices of different orders cannot be added or subtracted.

(ii) If $A = (a_{ij})_{m \times n}$, then

$-A = (-a_{ij})_{m \times n}$

1.1.23 Multiplication of Matrices

Let A and B be any two given matrices.

Then (i) The product 'AB' can be formed iff the number of columns of A is equal to the number of rows of B.

Then A, B are said to be conformable to the product AB.

(ii) The product 'BA' can be formed iff the number of columns of B is equal to the number of rows of A.

Then A, B are said to be conformable to the product BA.

Now we define the product AB as follows :

Let $A = (a_{ij})_{m \times n}$

and $B = (b_{ij})_{n \times p}$

The product AB can be formed, since number of columns (n) of A is equal to number of rows (n) of B.

Now $C = AB = (c_{ij})_{m \times p}$, where

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \ldots + a_{in} b_{nj} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad \ldots(1)$$

[Note that C = AB is an $m \times p$ matrix].

(1) is known as the inner product of i^{th} row of A with j^{th} column of B.

i.e., The general element c_{ij} of C is the sum of the products of corresponding elements of i^{th} row of A and those of j^{th} column of B.

Example :

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \end{bmatrix}$, and

$B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \\ -4 & 5 \end{bmatrix}$, find AB and BA.

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 0 \\ -4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (1.3+2.1+3.-4) & (1.-1+2.0+3.5) \\ (3.3+4.1+-1.-4) & (3.-1+4.0+-1.5) \end{bmatrix} = \begin{bmatrix} -7 & 14 \\ 17 & -8 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -1 \\ 1 & 0 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (3.1+-1.3) & (3.2+-1.4) & (3.3+-1.-1) \\ (1.1+0.3) & (1.2+0.4) & (1.3+0.-1) \\ (-4.1+5.3) & (-4.2+5.4) & (-4.3+5.-1) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 10 \\ 1 & 2 & 3 \\ 11 & 12 & -17 \end{bmatrix}$$

Note :(i) In the above example.

A is of order 2×3

B is of order 3×2

$\therefore$ AB is of order 2×2. BA is of order 3×3.

(ii) If A and B are two square matrices of same order, then both AB and BA can be found. Though they are of the same order, they need not be equal.

Some Special Matrices :

1.1.24 Idempotent Matrix

If A is a square matrix such that,

$A^2 = A \times A = A$, then A is said to be an idempotent matrix.

The unit matrix I is an idempotent matrix.

1.1.25 Nilpotent Matrix

If A is a square matrix such that $A^n = 0$ where n is a positive integer, then A is said to be nilpotent.

If n is the least positive integer satisfying the relation $A^n = 0$ then A is said to be nilpotent of order 'n' and n is called its index.

Example :

$A = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$ is nilpotent of order 2, since

$$A^2 = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

1.1.26 Periodic Matrices

If A is a square matrix such that $A^{n+1} = A$ where n is a positive integer then A is called a periodic matrix.

If 'n' is the least positive integer satisfying the relation $A^{n+1} = A$, then n is called the period of A.

Example :

(i) $$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}; A^2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$A^2 = A$; A is periodic of order one.

1.1.27 Involutary Matrix

If A is a square matrix such that $A^2 = I$ (I is unit matrix of order same as that of A), then A is said to be Involutary.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix} \text{ are involutary matrices.}$$

1.1.28 Unitary Matrix

If A is a square matrix with complex elements such that $A^* A = I$, then A is said to be unitary. (JNTU 2003)

Example :

If ω is a complex cube root of unity, then

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \text{ is a unitary matrix.[Solve example 7 after 1.1.30]}$$

1.1.29 Orthogonal Matrix

If A is a square matrix with real elements and $AA' = I = A'A$, then A is said to be orthogonal.

Example :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix}$$

$$AA' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & -\cos\theta \\ 0 & \cos\theta & \sin\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal.

Example :

Show that the matrix

$$A = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \text{ is orthogonal.}$$

$$A' = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$AA^{I} = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 + \frac{2}{\sqrt{6}}.\frac{2}{\sqrt{6}} + \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}} & 0.\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{6}}.\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}.\frac{-1}{\sqrt{3}} & 0.\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{6}}.\frac{-1}{\sqrt{6}} + \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}} \\ 0 + \frac{2}{6} - \frac{1}{3} & \frac{1}{2} + \frac{1}{6} + \frac{1}{3} & \frac{1}{2} - \frac{1}{6} - \frac{1}{3} \\ 0 - \frac{2}{6} + \frac{1}{3} & \frac{1}{2} - \frac{1}{6} - \frac{1}{3} & \frac{1}{2} + \frac{1}{6} + \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence A is orthogonal.

1.1.30 Properties of an Orthogonal Matrix

Let A be an orthogonal matrix.

Then 1. $AA' = I$

2. $\because AA^{-1} = I, A^{-1} = A'$
3. Product of two orthogonal matrices is an orthogonal matrix.
4. Transpose of an orthogonal matrix is an orthogonal matrix.
5. Inverse of an orthogonal matrix is orthogonal.
6. If A is an orthogonal matrix, $|A| = \pm 1$. ($|A|$ = determinant of A)
7. Any two row vectors or two column vectors of an orthogonal matrix are orthogonal, that is, their inner product is zero.

Solved Examples

EXAMPLE 1

Express $A = \begin{bmatrix} 1 & 6 & 10 \\ -3 & 4 & 5 \\ 6 & 1 & 7 \end{bmatrix}$ as the sum of an upper triangular matrix and a lower triangular matrix with diagonal elements equal to zero.

SOLUTION

Let A = B + C where (i) B is an upper triangular matrix and (ii) C is a lower triangular matrix with diagonal elements equal to zero.

Since A is 3×3 matrix, so are B and C.

Let $B = \begin{bmatrix} a & b & c \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 & 0 \\ l & 0 & 0 \\ m & n & 0 \end{bmatrix}$

$\because$ A = B + C, we have

$$\begin{bmatrix} 1 & 6 & 10 \\ -3 & 4 & 5 \\ 6 & 1 & 7 \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ l & 0 & 0 \\ m & n & 0 \end{bmatrix}$$

i.e., $$\begin{bmatrix} 1 & 6 & 10 \\ -3 & 4 & 5 \\ 6 & 1 & 7 \end{bmatrix} = \begin{bmatrix} a & b & c \\ l & p & q \\ m & n & r \end{bmatrix}$$

By equality of matrices, we have

$$a = 1, b = 6, c = 10, p = 4, q = 5, r = 7, l = -3, m = 6, n = 1.$$

$$\therefore \qquad B = \begin{bmatrix} 1 & 6 & 10 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 6 & 1 & 0 \end{bmatrix}$$

EXAMPLE 2

Prove that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a periodic matrix and its period is 4.

SOLUTION

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^4 = A^3.A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$A^5 = I_2.A = A$

$A^5 = A^{4+1} = A \Rightarrow$ A is periodic and of period 4.

EXAMPLE 3

Show that any square matrix can be expressed as the sum of a symmetric matrix and a skew symmetric matrix. (JNTU 2003)

SOLUTION

Let A be square matrix of order n . So A′ is also square matrix of order 'n'.

Let $B = \frac{1}{2}(A + A')$, and $C = \frac{1}{2}(A - A')$

so that B and C are square matrices of order n.

Now $B' = \left(\frac{1}{2}(A + A')\right)' = \frac{1}{2}(A + A')' \qquad \{\because (KA)' = KA'\}$

$= \frac{1}{2}(A' + (A')') \qquad \{\because (A+B)' = A' + B'\}$

$= \frac{1}{2}(A' + A) = B \Rightarrow$ B is symmetric.

Similarly

$$C^{l} = \frac{1}{2}(A'-A) = \frac{-1}{2}(A-A') = -C$$

$\Rightarrow$ C is skew symmetric

Again, $B + C = \frac{1}{2}[A + A' + A - A'] = A$

Hence the Problem.

EXAMPLE 4

Prove the uniqueness of the result of the above problem. (JNTU 2003)

SOLUTION

Let $A = P + Q$ where P is symmetric and Q is skew symmetric be another decomposition of A.

i.e., $P' = P,\ Q' = -Q$

Now $B = \frac{1}{2}(A + A') = \frac{1}{2}[P + Q + (P + Q)']$

$= \frac{1}{2}(P + P' + Q + Q')$

$= \frac{1}{2}(2P + 0) = P$; similarly $C = Q$

Hence the uniqueness of B and C.

EXAMPLE 5

Express $\begin{bmatrix} 3 & 7 \\ 4 & 5 \end{bmatrix}$ as sum of a symmetric and skew-symmetric matrices.

SOLUTION

Let $A = \begin{bmatrix} 3 & 7 \\ 4 & 5 \end{bmatrix}$ so that $A^{l} = \begin{bmatrix} 3 & 4 \\ 7 & 5 \end{bmatrix}$

$$A + A' = \begin{bmatrix} 6 & 11 \\ 11 & 10 \end{bmatrix};\ \text{let } B = \frac{1}{2}(A + A') = \begin{bmatrix} 3 & \frac{11}{2} \\ \frac{11}{2} & 5 \end{bmatrix}$$

$$A - A' = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix},\ \text{let } C = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & 3/2 \\ -3/2 & 0 \end{bmatrix}$$

B + C = A which completes the problem.

EXAMPLE 6

Determine the values of a, b, c such that $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is an orthogonal matrix.

SOLUTION

A is orthogonal $\Rightarrow$ AA' = I

$$\Rightarrow \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the elements of corresponding positions, we get,

$4b^2 + c^2 = 1$ .....(1)

$2b^2 - c^2 = 0$ .....(2)

$a^2 + b^2 + c^2 = 1$ .....(3)

$a^2 - b^2 - c^2 = 0$ .....(4)

(Deleting the repeated equalities)

$(1) + (2) \Rightarrow 6b^2 = 1 \Rightarrow b = \pm \frac{1}{\sqrt{6}}$

$(2) \Rightarrow c^2 = 2b^2 = \frac{1}{3} \Rightarrow c = \pm \frac{1}{\sqrt{3}}$

$(3) + (4) \Rightarrow 2a^2 = 1 \Rightarrow a = \pm \frac{1}{\sqrt{2}}$

$\therefore\ a = \pm \frac{1}{\sqrt{2}},\ b = \pm \frac{1}{\sqrt{6}},\ c = \pm \frac{1}{\sqrt{3}}$ are required values.

EXAMPLE 7

Show that $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ is a unitary matrix, where 'ω' is the complex cube root of unity.

SOLUTION

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

ω is cube root of unity ⇒ cube roots of unity are 1, ω, ω^2; where $\omega = \frac{-1+\sqrt{3}i}{2}$;

$$\omega^2 = \frac{-1-\sqrt{3}i}{2},$$

$$\therefore \ \bar{\omega} = \omega^2, \bar{\omega}^2 = \omega;$$

$$\bar{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \bar{\omega} & \bar{\omega}^2 \\ 1 & \bar{\omega}^2 & \bar{\omega} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$A^* = (\bar{A})' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$AA^* = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & (1+\omega^2+\omega) & (1+\omega+\omega^2) \\ (1+\omega+\omega^2) & (1+\omega^3+\omega^3) & (1+\omega^2+\omega^4) \\ (1+\omega^2+\omega) & (1+\omega^4+\omega^2) & (1+\omega^3+\omega^3) \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{pmatrix} \because \omega^4 = \omega, & \\ \omega^3 = 1 & \text{and} \\ 1+\omega+\omega^2 = 0 & \end{pmatrix}$$

$$= I_3$$

∴ A is unitary. [it can be proved that $A^*A = I_3$]

EXAMPLE 8

Show that $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$ is idempotent.

SOLUTION

$$A^2 = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} = \begin{bmatrix} 1+3-5 & -1-3+5 & 1+3-5 \\ -3-9+15 & 3+9-15 & -3-9+15 \\ -5-15+25 & 5+15+-25 & -5-15+25 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} = A \Rightarrow A \text{ is idempotent.}$$

EXAMPLE 9

Show that the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ is involutary

SOLUTION

$$A^2 = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0-12+12 & 3+9-12 & -3-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \text{ which shows that A is involutary.}$$

EXAMPLE 10

Show that $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of index 3.

SOLUTION

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0 \Rightarrow$ A is a nilpotent matrix of index 3.

Exercise - 1(a)

1. Express $A = \begin{bmatrix} 1 & 5 & 8 \\ -4 & 3 & 2 \\ 5 & 1 & 0 \end{bmatrix}$ as the sum of an upper triangular matrix and a lower triangular matrix with diagonal elements zero.

$$\left[\textbf{Ans:} \begin{pmatrix} 1 & 5 & 8 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix} \right]$$

2. Express the matrix $\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ as the sum of a symmetric matrix and a skew symmetric matrix.

$$\left[\textbf{Ans:} \begin{bmatrix} -1 & \frac{9}{2} & 3 \\ \frac{9}{2} & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & -2 \\ \frac{-5}{2} & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix} \right]$$

3. Show that the matrix $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is orthogonal.

4. Show that the matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

5. Show that, for $a \neq 0$, $b \neq 0$, the matrix $\begin{bmatrix} a & -b & -(a+b) \\ -a & b & (a+b) \\ a & -b & -(a+b) \end{bmatrix}$ is a nilpotent matrix of index 2.

6. Show that the matrix $\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi \\ -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi \end{bmatrix}$ is orthogonal.

7. Find a positive integer 'a' such that $\frac{1}{a}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix}$ is orthogonal. **(Ans : 3)**

8. Show that for any real values of a and b, the matrix $\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$ is nilpotent of index 2.

1.2 Rank of a Matrix

1.2.1 Submatrix

Let A be any given matrix. If some rows or columns or both of A are deleted from A, the resulting matrix is called a submatrix of A.

Ex.

Let $A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ -1 & 0 & 3 & 4 \\ 3 & -5 & 1 & -2 \end{bmatrix}$ be a given matrix of order 3 × 4.

(i) Deleting the 1st row of A, we get submatrix $\begin{bmatrix} -1 & 0 & 3 & 4 \\ 3 & -5 & 1 & -2 \end{bmatrix}$

(ii) Deleting the 1st column of A, we get the submatrix $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 4 \\ -5 & 1 & -2 \end{bmatrix}$

(iii) Deleting the 3rd and 4th columns of A, we get the submatrix $\begin{bmatrix} 4 & 3 \\ -1 & 0 \\ 3 & -5 \end{bmatrix}$

(iv) If no row and no column are deleted, A remains unchanged. So A is also a submatrix of A.

1.2.2 Minors of Matrix

The determinant of a square submatrix of a given matrix is called its minor. If the order of the square submatrix is 'r' then the corresponding minor is said to be an 'r-rowed minor' or a 'minor of order r'.

Ex.

Let $A = \begin{bmatrix} 1 & -2 & 3 & -4 \\ 0 & 5 & -1 & 2 \\ 3 & -3 & 6 & -2 \end{bmatrix}$ be a 3 × 4 matrix. Then,

(i) $\begin{vmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \\ 3 & -3 & 6 \end{vmatrix}, \begin{vmatrix} -2 & 3 & -4 \\ 5 & -1 & 2 \\ -3 & 6 & -2 \end{vmatrix}, \begin{vmatrix} 1 & -2 & -4 \\ 0 & 5 & 2 \\ 3 & 3 & -2 \end{vmatrix},$

are minors of order 3 or 3 - rowed minors of A.

(ii) $\begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix}$, $\begin{vmatrix} -2 & -4 \\ -3 & -2 \end{vmatrix}$, $\begin{vmatrix} 0 & -1 \\ 3 & 6 \end{vmatrix}$, ... are 2-rowed minors.

(iii) Each element of A can be regarded as a 1-rowed minor of A.

(iv) 'A' does not have a 4-rowed minor as it does not have a 4×4 submatrix.

1.2.3 Rank of a Matrix

Let A be a given matrix of order $m \times n$.

Let $r \geq 0$ be a positive integer such that (i) There exists at least one non-zero minor of order 'r'and (ii) All minors of higher orders vanish.

Then 'r' is called the rank of 'A' and is generally denoted by $\rho(A)$, or R(A) or r(A). We denote it by $\rho(A)$.

(or) Briefly, the rank of a matrix can be defined as the order of the largest non vanishing minor of the matrix.

Note : (i) If A is a matrix of order $m \times n$, then $\rho(A) \leq m$ and $\rho(A) \leq n$; (i.e., $\rho(A) \leq$ minimum of m and n)

(ii) If A is a null matrix, $\rho(A) = 0$ since all minors of A vanish.

(iii) If A is a square matrix of order 'n', then $\rho(A) \leq n$.

(iv) If A is a singular matrix of order 'n', then $|A| = 0 \Rightarrow \rho(A) < n$.

(v) If A is a non singular matrix of order n, then $|A| \neq 0 \Rightarrow \rho(A) = n$. In particular, $\rho(I_n) = n$.

(vi) $\rho(A') = \rho(A)$.

1.2.4 Calculation of the rank of a given matrix

Let A be given matrix . To find $\rho(A)$, the following procedure can be followed.

Step 1 : Find the largest possible minor of A. If this minor is not equal to zero, its order is the rank of A. If all the minors of the largest possible order vanish, then step 2.

Step 2 : Find the minors of the next lower order. If any one them is not equal to zero, its order is equal to the rank of A. However, if all these minors too vanish, find the minors of next lower order and so on.

Repeat this process till a non-vanishing minor of the largest order is obtained.

Solved Examples

EXAMPLE 11

Find the ranks of the following matrices :

(i) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

SOLUTION

Let the given matrix be 'A'.

(i) $|A| = 0 \Rightarrow \rho(A) \neq 3.$

We find minors of order 2.

$$\begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} = 0, \begin{vmatrix} -2 & 3 \\ -4 & 6 \end{vmatrix} = 0, \begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} = 0, ...$$

We find all minors of order 2 vanish.

$\therefore \quad \rho(A) \neq 2;$

A is a non zero matrix $\Rightarrow \rho(A) \neq 0$

$\therefore \quad \rho(A) = 1$

(ii) $|A| = 0 \Rightarrow \rho(A) \neq 3.$

There exist a minor $\begin{vmatrix} 2 & -2 \\ 3 & 4 \end{vmatrix}$ which is not equal to zero.

$\therefore \quad \rho(A) = 2$

(iii) $|A| = 1(3-2) - 2(2-3) + 3(4-9) = 1 + 2 - 15 = -12 \neq 0.$

$\therefore \quad \rho(A) = 3$

(iv) $|A| = 0 \Rightarrow \rho(A) \neq 3$

All 2^{nd} order minor vanish,

$\therefore \quad \rho(A) \neq 2;$

A is a non-zero matrix.

$\therefore \quad \rho(A) = 1$

EXAMPLE 12

Find the value of 'a' such that the rank of the matrix $\begin{bmatrix} 3 & 5 & a \\ 2 & 1 & -1 \\ 1 & 4 & 2 \end{bmatrix}$ is 2.

SOLUTION

Let $\quad A = \begin{bmatrix} 3 & 5 & a \\ 2 & 1 & -1 \\ 1 & 4 & 2 \end{bmatrix}$

Since $\rho(A) = 2$, $|A| = 0$.

i.e., $\quad 3(2+4) - 5(4+1) + a(8-1) = 0$

$\quad 18 - 25 + 7a = 0 \Rightarrow a = 1$

Exercise 1(b)

Find the ranks of the following matrices.

1. $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 0 & 3 & 1 \end{bmatrix}$ **(Ans :** 3)

2. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & 3 & 6 \end{bmatrix}$ **(Ans :** 2)

3. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ **(Ans :** 1)

4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **(Ans :** 3)

5. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix}$ **(Ans :** 1)

6. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **(Ans :** 2)

7. If the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \\ 2 & b & 4 \end{bmatrix}$ is 2, find 'b'. **(Ans :** $\frac{11}{4}$)

1.2.5 Zero rows and non-zero rows

If each element of a row is zero, it is called a zero-row.

If at least one element of a row is non-zero, it is called a non-zero row.

Example :

In matrix $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, 3^{rd} row is a zero row; whereas 1^{st} and 2^{nd} rows are non-zero rows.

1.2.6 Elementary Transformations or Elementary Operations

Elementary row transformations :

1. The interchange of i^{th} row and j^{th} row of a matrix will be denoted by 'R_{ij}' or $R_i \leftrightarrow R_j$
2. The multiplication of the elements of i^{th} row of a matrix by a non-zero number 'l' will be denoted by '$R_{i(l)}$ or $R_i \rightarrow l \,.\, R_i$
3. The addition of 'l' times the elements of the j^{th} row to the corresponding elements of i^{th} row of a matrix will be denoted by '$R_{ij(l)}$' or $R_i \rightarrow R_i + l \,.\, R_j$.

Elementary column transformations can be done in a similar manner writing C in place of R.

1.2.7 Elementary Matrices

A matrix obtained by operating an elementary transformation on a unit matrix is called an elementary matrix (or E - matrix).

Example :

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ are E - matrices obtained from operating respectively the operations.

R_{13} ($R_1 \Leftrightarrow R_3$), $R_{2(3)}$($R_2 \rightarrow 3R_2$), and $R_{23(4)}$ ($R_2 \rightarrow R_2 + 4R_3$) on I_3.

1.2.8 Equivalent Matrices

Two matrices A and B of same order are said to be equivalent matrices if one of them is obtained from the other by elementary transformations. Then we write '$A \sim B$'.

1.2.9 Methods to determine the rank of a matrix

The method given in 1.2.4 to find the rank is applicable if the matrix is of a smaller order. But if the order of the matrix is not small, the process becomes laborious since we have to calculate many minors of higher orders. Hence a few alternative methods are given in the coming sections.

1.2.10 Echelon form of a matrix

A given matrix is said to be in Echelon form if :

1. The zero rows of the matrix, if they exist, occur below the non-zero rows of the matrix.
2. The first non-zero element in any non-zero row of the matrix must be equal to unity.
3. The number of zeroes before the first non-zero element in any non-zero row is less than the number of such zeroes in the next non-zero row.

   (**Note :** Condition (2) is not followed by some authors).

Example :

$$1.\ \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 2.\ \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 3.\ \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are examples of Echelon form :

Note :

1. The rank of a matrix is unaltered by elementary transformations.
2. The rank of a matrix in the Echelon form is equal to the number of non-zero rows in it.
3. To find the rank, we therefore reduce the given matrix into an Echelon form by using elementary transformations.

1.2.11 Normal Form or Canonical Form

(i) A matrix of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where I_r is a unit matrix of order 'r' and 0 is the null matrix is called the normal form or first canonical form.

(ii) Matrices of the form $\begin{bmatrix} I_r & 0 \end{bmatrix}$, $\begin{bmatrix} I_r \end{bmatrix}$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, are also known as normal forms.

(iii) Every $m \times n$ matrix can be reduced to the normal form by a series of elementary transformations.

(iv) The rank of the above canonical form is 'r'.

(v) *Algorithm to reduce a matrix to the normal form :*

Step 1 : Make $a_{11} = 1$, by applying suitable row or column operations or both.

Step 2 : Make all the elements below a_{11} in 1st column and on the right of a_{11}, in the 1st row equal to zero with the help of suitable row and column operations.

Step 3 : If the resulting matrix is in the normal form, the process ends. Otherwise repeat the above steps regarding a_{22} such that no row or column operation should involve 1st row or 1st column. If the matrix is still not in the normal form proceed to a_{33} and apply the same process. The process is continued till the normal form is obtained.

Solved Examples

EXAMPLE 13

Reduce the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 & 6 \\ 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 8 \\ 1 & 1 & 2 & 2 \end{bmatrix} \text{ to an Echelon form and hence find its rank.}$$

SOLUTION

Step 1 : To get '1' in the first position of 1st row, (i.e., a_{11} position).

Applying R14 ($R1 \leftrightarrow R_4$), we get

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 8 \\ 3 & 1 & 4 & 6 \end{bmatrix}$$

Step 2 : To get zeros below a_{11}.

Applying $R_2 \to R_2 - 2R_i$; $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 3R_1$,

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

Step 3 : To get 1 in a_{22} position.

Applying $R_2 \rightarrow -R_2$,

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

Step 4 : To get zeros below a_{22} :

Applying $R_3 \rightarrow R_3 + 2R_2$, $R_4 \rightarrow R_4 + 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Step 5 : $a_{33} = 1$; so we have to get zeros below it. $\therefore$ Applying $R_4 \rightarrow R_4 - 2R_3$;

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$, which is in the Echelon form which has 3 non-zero rows in it.

$\therefore \qquad \rho(A) = 3$

EXAMPLE 14

Apply elementary transformations to find the rank of

$$A = \begin{bmatrix} 1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix}$$

SOLUTION

Applying $R_2 \to R_2 - 7R_1$; $R_3 \to R_3 - 5R_1$, we get,

$$A \sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 69 & -23 & 46 \\ 0 & 33 & -11 & 22 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 3 & -1 & 2 \\ 0 & 3 & -1 & 2 \end{bmatrix} \quad \left(R_2 \to \frac{1}{23}R_2; R_3 \to \frac{1}{11}R_3\right)$$

$$\sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B \text{ (say) } [R_3 \to R_3 - R_2],$$

which is in Echelon form which has two non-zero rows,

$\therefore \quad \rho(A) = \rho(B) = 2$

(or) every 3-rowed minor of B = 0

and $\begin{vmatrix} 1 & -7 \\ 0 & 3 \end{vmatrix} \neq 0.$

$\therefore \quad \rho(A) = 2$

EXAMPLE 15

Find the rank of the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$,

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \begin{Bmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} (R_4 \to R_4 - (R_2 + R_3))$$

= B ; In B, the 4 rowed minor = 0

$\therefore \quad \rho(A) \neq 4$

The leading 3 rowed minor,

$$= \begin{vmatrix} 1 & -1 & -2 \\ 0 & 5 & 3 \\ 0 & 4 & 9 \end{vmatrix} = 45 - 12 \neq 0$$

$\therefore \quad \rho(A) = 3.$

(**Note :** The problem can also be solved by reducing to Echelon form).

EXAMPLE 16

Find the constants 'l' and 'm' such that the rank of the matrix $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & l & m \end{bmatrix}$ is (i) 3 (ii) 2.

SOLUTION

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & l & m \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 10 & (l-18) & (m-6) \end{bmatrix} \begin{Bmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 6R_1 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & (l-4) & (m-6) \end{bmatrix} (R_3 \to R_3 - 2R_2) = B$$

Matrix B is in Echelon form.

(i) If $\rho(A) = 3$, then $\rho(B) = 3$, $\therefore$ B should have 3 non zero rows.

$\therefore \quad l \neq 4$ (or) $m \neq 6$.

(ii) If $\rho(A) = 2$, then $\rho(B) = 2$, $\therefore$ B should have 2 non-zero rows.

$\therefore \quad l = 4$ and $m = 6$.

EXAMPLE 17

Find the rank of the matrix $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 3 & 4 & 8 & 11 \\ 1 & 3 & 6 & 7 \end{bmatrix}$.

SOLUTION

The given matrix be A. Then,

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 3 & 4 & 8 & 11 \\ 1 & 3 & 6 & 7 \end{bmatrix} (R_1 \leftrightarrow R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 4 & 4 \end{bmatrix}, \begin{Bmatrix} R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix} = B \text{ (say)}$$

B is in Echelon form with two non zero rows.

$\therefore \quad \rho(A) = \rho(B) = 2.$

EXAMPLE 18

Reduce the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 3 & 4 & 5 \end{bmatrix}$ to normal form and hence find its rank.

SOLUTION

Given matrix be A. Then,

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix}, \begin{Bmatrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix}, \begin{Bmatrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 3C_1 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{Bmatrix} R_2 \to R_2 \times -\frac{1}{2} \\ R_3 \to R_3 - R_2 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, (C_3 \to C_3 - 2C_2),$$

which is of the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence $\rho(A) = 2.$

EXAMPLE 19

Reduce the matrix $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \end{bmatrix}$ into the normal form and hence find its rank.

SOLUTION

Given matrix be A. Then

$$A \sim \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 2 & -2 & 2 \\ 0 & 10 & -4 & -6 \\ 0 & 1 & -1 & 1 \end{bmatrix}, \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 10 & -4 & -6 \\ 0 & 1 & -1 & 1 \end{bmatrix}, \begin{matrix} C_2 = C_2 + 2C_1 \\ C_3 = C_3 - C_1 \cdot \\ C_4 = C_4 - 2C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 10 & -4 & -6 \\ 0 & 1 & -1 & 1 \end{bmatrix} R_2 \to \frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 6 & -6 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} C_2 \to C_2 + C_3 + C_4 \\ C_3 \to C_3 + C_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} R_3 \to \frac{1}{6} - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} C_4 \to C_4 - C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \rightarrow R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} C_4 \rightarrow C_4 + C_3$$

which is of the normal form $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $\rho(A) = 3$.

EXAMPLE 20

Reduce the matrix $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 1 & 3 & 2 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$, to normal form and find its rank.

SOLUTION

Given matrix be A. Then

$$A \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 3 & 5 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix}, \begin{array}{l} R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 5 \\ 0 & 3 & 5 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix}, \begin{array}{l} C_3 \rightarrow C_3 + 3C_1 \\ C_4 \rightarrow C_4 - 2C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & -3 & -7 \end{bmatrix}, \begin{array}{l} R_3 \rightarrow R_3 - 3R_2 \\ R_4 \rightarrow R_4 - R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & -3 & -7 \end{bmatrix}, \begin{matrix} C_3 & \to & C_3 - 4C_2 \\ C_4 & \to & C_4 - 5C_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{17}{7} \\ 0 & 0 & -3 & -7 \end{bmatrix}, R_3 \to \frac{-1}{7} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{17}{7} \\ 0 & 0 & 0 & \frac{2}{7} \end{bmatrix}, R_4 \to R_4 + 3R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{2}{7} \end{bmatrix}, C_4 \to C_4 - \frac{17}{7} C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R_4 \to \frac{7}{2} R_4$$

which is in the normal form I_4. $\therefore\ \rho(A) = 4$

EXAMPLE 21

If $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix}$, find two non-singular matrices P and Q such that PAQ is in the normal form.

Method of finding two non-singular matrices P and Q such that PAQ is in the normal form where A is any given m × n matrix.

Step 1 :

Write $A = I_m A I_n$, ...(1)

where I_m, I_n are unit matrices of orders m and n respectively.

Step 2 :

Subject the matrix A on the L.H.S of (1) to elementary row and column transformations to reduce it to the normal form. Perform each of the same row transformations on I_m of R.H.S and each of the same column transformations on I_n of R.H.S. In all the steps keep the matrix A on R.H.S as it is.

Solution :

Given matrix be A. Since it is of order 3 × 4, we write

$A = I_3 A I_4$

i.e., $$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ Applying

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 2R_1$ on L.H.S as well as the prefactor (I_3) of A on R.H.S, we get,

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 6 & 6 & -9 \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 + C_1$, $C_3 \to C_3 + C_1$, $C_4 \to C_4 - 2C_1$, on L.H.S as well as the postfactor (I_4) of A on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 6 & -9 \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to \frac{1}{6} R_2$ on L.H.S as well as the prefactor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{-3}{2} \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{6} & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 4R_2$ on L.H.S as well as prefactor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{-3}{2} \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{6} & 0 \\ \frac{2}{3} & \frac{-2}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_3 \to C_3 - C_2$ on L.H.S as well as the post factor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{2} \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{6} & 0 \\ \frac{2}{3} & \frac{-2}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to \frac{-1}{2} R_3$ on L.H.S as well as the prefactor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{2} & \frac{1}{6} & 0 \\ \frac{-1}{3} & \frac{1}{3} & \frac{-1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_4 \to C_4 + \left(\frac{3}{2}\right) C_2$ on L.H.S as well as post factor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{6} & 0 \\ \frac{-1}{3} & \frac{1}{3} & \frac{-1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & \frac{-1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The L.H.S which is now of the form $[I_3\ 0]$ is in the normal form. Hence the two required matrices are

$$P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{6} & 0 \\ \frac{-1}{3} & \frac{1}{3} & \frac{-1}{2} \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 1 & 0 & \frac{-1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE 22

Find two non-singular matrices P and Q such that P AQ will be in the normal form where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$$

SOLUTION

Here A is 3×3 matrix, hence we write $A = I_3 A I_3$

i.e., $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ on L.H.S as well as the prefactor of A on R.H.S, we get,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -6 \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$ on L.H.S as well as the postfactor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & -6 \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to \frac{-1}{5} R_2$ on L.H.S as well as the prefactor of R.H.S, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{6}{5} \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{-1}{5} & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 + 5R_2$ on L.H.S and the prefactor on R.H.S, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{6}{5} \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{-1}{5} & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_3 \to C_3 - \frac{6}{5}C_2$, on L.H.S and the postfactor on R.H.S, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{-1}{5} & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & \frac{-3}{5} \\ 0 & 1 & \frac{-6}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to -R_3$ on L.H.S and the prefactor of R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{-1}{5} & 0 \\ 1 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & \frac{-3}{5} \\ 0 & 1 & \frac{-6}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Now the L.H.S is in the normal form (I_3). Hence the required matrices are

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 2/5 & -1/5 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -2 & \frac{-3}{5} \\ 0 & 1 & \frac{-6}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 1(c)

I. Reduce the following matrices to Echelon form and hence find their ranks.

(a) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ **[Ans : 3]**

(b) $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & -3 & 1 & 4 \\ 3 & 4 & -2 & 6 \end{bmatrix}$ **[Ans : 3]**

(c) $\begin{bmatrix} 3 & 4 & 5 & 2 \\ -1 & 2 & 0 & -3 \\ 1 & -1 & 1 & 4 \end{bmatrix}$ **[Ans : 3]**

(d) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ **[Ans : 2]**

II Reduce the following matrices to the normal form and hence find their ranks.

(a) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **[Ans : 2]**

(b) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}$ **[Ans : 2]**

(c) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ **[Ans : 2]**

(d) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & -4 \end{bmatrix}$ **[Ans : 3]**

III Find two non-singular matrices P and Q such that P AQ is in the normal form where A is given by the matrix.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$ **Ans :** $P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 3 & -2 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ **Ans :** $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{-1}{3} & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

IV Find the constants 'a' and 'b' such that the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 2 & -3 \\ 1 & 1 & a & b \end{bmatrix}$ **is of rank**

(i) 3 and (ii) 2.

Ans : (i) $a \neq \frac{1}{2}$ **(or)** $b \neq \frac{7}{2}$

(ii) $a = \frac{1}{2}$ and $b = \frac{7}{2}$

V If the rank of the matrix $\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -1 \\ 3 & 0 & a \end{bmatrix}$ **is '2', find the value of 'a'.**

[Ans : 2]

1.3 Inverse of a Square Matrix-Gauss Jardan Method

1.3.1 Singular and non-singular matrices

Let A be a square matrix.

(i) If $|A| = 0$, A is said to be singular.

(ii) If $|A| \neq 0$, A is said to be non-singular.

Example :

(i) $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ is singular, $\because \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 4 - 4 = 0$

(ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 6 \\ 2 & 3 & 4 \end{bmatrix}$ is non singular, $\because$ det $A = -2 - 1 = -3 \ (\neq 0)$

1.3.2 Inverse of a square matrix

(i) Let A be a non singular matrix of order 'n'.

(ii) Let there exist a matrix B of same order such that $AB = BA = I_n$ where I_n is unit matrix of order n.

Then B is called the inverse of A and is denoted by A^{-1}.

Example :

Let $A = \begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix}$ $\left(\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \neq 0 \therefore \text{A is non singular} \right)$

Let $B = \frac{1}{3}\begin{bmatrix} 6 & -3 \\ -1 & 1 \end{bmatrix}$

$$AB = BA = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Hence $A^{-1} = B = \frac{1}{3}\begin{bmatrix} 6 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1/3 & 1/3 \end{bmatrix}$

Note : (i) Rectangular matrices and singular matrices do not have inverses.

(ii) A square matrix which possesses an inverse is called an invertible matrix.

(iii) A square matrix, if invertible, has one and only one inverse, that is, it has an unique inverse.

1.3.3 Computation of inverse of a given matrix by Gauss-Jordan method

Gauss-Jordan method is one of the methods of finding the inverse. In this method we use the elementary row (or column) transformations. The method is given below:

1st method :

Let 'A' be a given non-singular matrix of order n. Let I_n be the unit matrix of order 'n'.

Step 1 :

Write $A = I_n A$ .....(1)

Step 2 :

Apply a chain of elementary row transformations on L.H.S and the prefactor I_n of A on R.H.S of (1) till the equation (1) is reduced to the form

$$I_n = BA$$

Then $B = A^{-1}$

(or) 2nd method :

Write $A = AI_n$ and reduce it to the form $I_n = AB$ with the help of a chain of elementary column transformations, so that $B = A^{-1}$.

(or) 3rd method :

Take the matrix $\left[A \;\vdots\; I_n\right]$ and reduce it with the help of a series of row operations to the form $\left[I_n \;\vdots\; B\right]$. Then $B = A^{-1}$.

Solved Examples

EXAMPLE 23

Compute the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & -2 \\ 2 & 0 & -4 \end{bmatrix}$ by Gauss-Jordan method.

SOLUTION

Let $A = I_3A$

i.e., $$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & -2 \\ 2 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad .....(1)$$

We use row transformations on the L.H.S as well as I_3 on R.H.S of (1)

$R_2 \to R_2 + R_1$, and $R_3 \to R_3 - 2R_1$ give

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & -4 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

$R_2 \to \frac{1}{5} R_2$ gives,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{5} \\ 0 & -4 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$, and $R_3 \to R_3 + 4R_2$ give

$$\begin{bmatrix} 1 & 0 & \frac{13}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & \frac{-46}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{-2}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{-6}{5} & \frac{4}{5} & 1 \end{bmatrix} A$$

$R_3 \to \frac{-5}{46} R_3$ gives,

$$\begin{bmatrix} 1 & 0 & \frac{13}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{-2}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{3}{23} & \frac{-2}{23} & \frac{-5}{46} \end{bmatrix} A$$

$R_1 \to R_1 - \frac{13}{5} R_3$, and $R_2 \to R_2 - \frac{1}{5} R_3$ give

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{23} & \frac{-4}{23} & \frac{13}{46} \\ \frac{4}{23} & \frac{5}{23} & \frac{1}{46} \\ \frac{3}{23} & \frac{-2}{23} & \frac{-5}{46} \end{bmatrix} A$$

which is of the form $I_3 = BA$

Hence $$A^{-1} = B = \begin{bmatrix} \frac{6}{23} & \frac{-4}{23} & \frac{13}{46} \\ \frac{4}{23} & \frac{5}{23} & \frac{1}{46} \\ \frac{3}{23} & \frac{-2}{23} & \frac{-5}{46} \end{bmatrix} = \frac{1}{46}\begin{bmatrix} 12 & -8 & 13 \\ 8 & 10 & 1 \\ 6 & -4 & -5 \end{bmatrix}$$

EXAMPLE 24

If $A = \begin{bmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, find A^{-1}

SOLUTION

Let $A = AI_3$

i.e., $$\begin{bmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ......(1)$$

We use column transformations on L.H.S as well as on I_3 of R.H.S of (1),

$C_1 \leftrightarrow C_3$ gives,

$$\begin{bmatrix} 1 & -1 & 4 \\ -1 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 + C_1$ and $C_3 \to C_3 - 4C_1$ give

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 6 \\ 3 & 2 & -11 \end{bmatrix} = A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

$C_2 \to -C_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 6 \\ 3 & -2 & -11 \end{bmatrix} = A \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & -4 \end{bmatrix}$$

$C_1 \to C_1 + C_2$, and $C_3 \to C_3 - 6C_2$ give

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 6 \\ 0 & -1 & 2 \end{bmatrix}$$

$C_1 \to C_1 - C_3$ and $C_2 \to C_2 + 2C_3$ give

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -1 & 2 & 1 \\ -7 & 11 & 6 \\ -2 & 3 & 2 \end{bmatrix}$$

which is of the form

$$I_3 = AB$$

Hence $$A^{-1} = B = \begin{bmatrix} -1 & 2 & 1 \\ -7 & 11 & 6 \\ -2 & 3 & 2 \end{bmatrix}$$

EXAMPLE 25

Find A^{-1} when $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 1 \\ -3 & 2 & 1 \end{bmatrix}$

SOLUTION

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] (R_1 \leftrightarrow R_2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 5 & 2 & 1 & -2 & 0 \\ 0 & -1 & 4 & 0 & 3 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2/5 & 1/5 & -2/5 & 0 \\ 0 & -1 & 4 & 0 & 3 & 1 \end{array}\right] R_2 \to \frac{1}{5} R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 7/5 & 1/5 & 3/5 & 0 \\ 0 & 1 & 2/5 & 1/5 & -2/5 & 0 \\ 0 & 0 & \frac{22}{5} & 1/5 & 13/5 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 7/5 & 1/5 & 3/5 & 0 \\ 0 & 1 & 2/5 & 1/5 & -2/5 & 0 \\ 0 & 0 & 1 & 1/22 & 13/22 & 5/22 \end{array}\right] R_3 \to \frac{5}{22} R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/22 & \frac{-5}{22} & \frac{-7}{22} \\ 0 & 1 & 0 & 2/11 & -7/11 & -1/11 \\ 0 & 0 & 1 & 1/22 & 13/22 & 5/22 \end{array}\right] \begin{array}{l} R_1 \to R_1 - \frac{7}{5} R_3 \\ R_2 \to R_2 - \frac{2}{5} R_3 \end{array}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{3}{22} & \frac{-5}{22} & \frac{-7}{22} \\ \frac{2}{11} & \frac{-7}{11} & \frac{-1}{11} \\ \frac{1}{22} & \frac{13}{22} & \frac{5}{22} \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 3 & -5 & -7 \\ 4 & -14 & -2 \\ 1 & 13 & 5 \end{bmatrix}$$

Exercise 1(d)

I. Find the inverses of the following matrices by Gauss-Jordan method.

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & -4 & 5 \end{bmatrix}$ **Ans :** $\frac{1}{16}\begin{bmatrix} -31 & 22 & 1 \\ -2 & 4 & 2 \\ 17 & -10 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 4 & -2 \\ 2 & 1 & 0 \\ 1 & -1 & 5 \end{bmatrix}$ **Ans :** $\frac{1}{19}\begin{bmatrix} -5 & 18 & -2 \\ 10 & -17 & 4 \\ 3 & -7 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 2 & 1 \\ -1 & -2 & 3 \\ 5 & 1 & 2 \end{bmatrix}$ **Ans :** $\frac{1}{15}\begin{bmatrix} -7 & -3 & 8 \\ 17 & 3 & -13 \\ 9 & 6 & -6 \end{bmatrix}$

1.4 Simultaneous Linear Equations

In this section, the theory of matrices is applied to determine the existence of solutions of simultaneous linear equations and obtain the solutions if they exist.

1.4.1 Matrix form of a system of nonhomogeneous Linear Equations

A system of m simultaneous nonhomogeneous linear equations in n unknowns $x_1, x_2, x_3, ...x_n$ is of the form

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots a_{mn}x_n = b_m \end{matrix}\right\} \quad .....(1)$$

The above system can be written in a matrix form (using the concepts of multiplication of matrices and equality of matrices) as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ ; \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ ; \\ b_m \end{bmatrix} \quad .....(2)$$

or simply, $AX = B$ .....(3)

where $B \neq 0$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$, which is called the coefficient matrix,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$ which is the solution matrix

and $$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$ is the matrix of the constants $b_1, b_2, \ldots b_m$ of the R.H.S of (1)

The set of values $x_1, x_2, \ldots x_n$ which satisfy the system (1) is called the solution of the system.

1.4.2 Consistency and Inconsistency

The system of equations (1) is said to be consistent if it has at least one solution.

(1) is said to be inconsistent if it has no solution.

1.4.3 Augmented matrix (A/B)

The m × (n + 1) matrix given by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} & b_2 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} & b_m \end{bmatrix}$$

(which is obtained by annexing the elements of B to those of (A) is called the augmented matrix and is denoted by (A/B) or (A, B).

1.4.4 Condition for the consistency of (1)

1. The condition for the consistency of (1) is given by the following theorem (with out proof).
2. ***Theorem*** : The system of simultaneous linear equations AX = B is consistent if and only if the rank of A = rank of (A/B).

Note : 1. Thus the ranks of the coefficient matrix and the augmented matrix play a vital role in determining the consistency of the system (1).

2. We apply the elementary transformations to the augmented matrix (A/B) since they do not alter its rank. This reduced equivalent system will enable us to test for consistency and also to find the solution.

1.4.5 Working Rules

Step 1 : Write the system in the form AX = B.

Step 2 : Find $\rho(A)$ and $\rho(A/B)$ by applying elementary transformations on (A/B).

Conclusions :

Case (i) : If $\rho(A) = \rho(A/B) = n$, where 'n' is the number of unknowns, the system is consistent and it has a unique solution [in this case A is non-singular]. This unique solution (given by $X = A^{-1} B$) can be obtained by the reduced system of equations.

Case (ii) : If $\rho(A) = \rho(A/B) < n$, then the system is consistent and has an infinite number of solutions.

Case (iii) : If $\rho(A) \neq \rho(A/B)$, the system is inconsistent and has no solution.

Note : $\rho(A) \not> \rho(A/B)$

Summary :

The nature of solutions of the system AX = B with respect to its consistency or inconsistency is shown in the following table.

Consistent	$\rho(A, B) = \rho(A) = n$	unique solution
	$\rho(A, B) = \rho(A) < n$	Infinite solutions
Inconsistent	$\rho(A, B) \neq \rho(A)$	No solution
	i.e, $\rho(A, B) > \rho(A)$	

1.4.6 Some Other Methods of Solving Non-homogenous Linear Equations

In this section, certain methods of solving non-homogeneous equations (in which the concept of the rank of a matrix need not be used) are discussed.

Note that these are applicable only when the coefficient matrix of the system is a square matrix.

Method 1 : *Matrix inversion method :*

The solution of the system AX = B is given by $X = A^{-1}B$ provided A^{-1} exists.

∴ This method is applicable only when A is nonsingular.

Method 2 : *CRAMER'S RULE (using determinants) :*

To apply this method A must be non-singular.

i.e., $|A| \neq 0$

Let A be 3×3 matrix.

Let $|A| = \Delta$

we find the values of 3 other determinants which are obtained by replacing the 1st, 2nd and 3rd columns of Δ with the column matrix 'B' of the given system respectively.

Let $\Delta_1, \Delta_2, \Delta_3$ be the values of these determinants respectively.

Then, $x = \dfrac{\Delta_1}{\Delta}, y = \dfrac{\Delta_2}{\Delta}, z = \dfrac{\Delta_3}{\Delta}$

Method 3 : *Gauss - Jordan method :*

Take the matrix (A, B) and subject it to a series of elementary row transformations till it is reduced to the form $(I_n X)$ where I_n is unit matrix of order n and X is the column matrix of the order of B. Then 'X' is the solution matrix.

1.4.7 Gauss Elimination Method

This method consists of two steps. The first step is the elimination of unknowns. The second step is to get the values of unknowns by back substitution.

Method :

Consider the system of equations,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad \ldots\ldots (1)$$

Step 1 : By elminating the unknowns $x_1, x_2, \ldots, x_{n-1}$, we obtain an upper-triangular system.

For example, x_1 can be eliminated by multiplying the first equation of (1) with $\left(\frac{-a_{21}}{a_{11}}\right)$ and adding it to the second equation of (1) [Note that $a_{11} \neq 0$]. Similarly, multiply the 1st equation by $\left(\frac{-a_{31}}{a_{11}}\right)$ and add it to 3rd equation, multiply 4th equation with $\left(\frac{-a_{41}}{a_{11}}\right)$ and add to 4th equation and so on. By this process, x_1 is eliminated from 2nd, 3rd, 4th, . . . and nth equations. Thus the system (1) is reduced to,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a'_{21}x_1 + a'_{22}x_2 + \ldots + a'_{2n}x_n &= b'_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a'_{n1}x_1 + a'_{n2}x_2 + \ldots + a'_{nn}x_n &= b'_n \end{aligned}\right\} \quad \ldots\ldots (2)$$

where $a'_{22} = a_{22} - a_{12}\frac{a_{21}}{a_{11}}$

$$a'_{32} = a_{32} - a_{12}\frac{a_{31}}{a_{11}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$b'_2 = b_2 - b_1\frac{a_{21}}{a_{11}}$$

$$b'_3 = b_3 - b_1\frac{a_{31}}{a_{11}}$$

$\ldots\ldots\ldots\ldots\ldots\ldots$, and so on.

The same process of elimination is used successively till we arrive at an upper-triangular system as given below,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + \ldots + a'_{2n}x_n &= b'_2 \\ a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ + a_{nn}^{(n-1)}x_n &= b_n^{(n-1)} \end{aligned}\right\} \quad \ldots\ldots (3)$$

[Note that the primes indicate the number of changes that the element has undergone. For example, $a''_{33} \Rightarrow a_{33}$ has changed twice and $a_{nn}^{(n-1)} \Rightarrow a_{nn}$ has changed (n – 1) times]

Step 2 : Now we get the value of x_n from last equation of (3).

Back substituting x_n in the last but one equation, we get the value of x_{n-1} and so on.

1.4.8 LU Decomposition

Consider a system of linear equations,

$$AX = B \quad \ldots\ldots (1)$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Further, let all the principal minors of A be non-singular. i.e., $a_{11} \neq 0$,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \text{ and } |A| \neq 0$$

then we can write,

$$A = LU \quad \ldots\ldots (2)$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ ..... (3)

is a lower triangular matrix, and

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \ldots\ldots (4)$$

is an upper triangular matrix, then (1) and (2) $\Rightarrow$,

$$LUX = B \quad \text{..... (5)}$$

$$\Rightarrow \quad LY = B \quad \text{..... (6)}$$

where $$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = UX \quad \text{..... (7)}$$

$$(6) \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3, \end{aligned}$$

from which y_1, y_2, y_3 can be found by forward substitution. Thus Y is found.

then (7) $\Rightarrow$ $UX = Y$

$$\Rightarrow \quad \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1, \\ u_{22}x_2 + u_{23}x_3 &= y_2, \\ u_{33}x_3 &= y_3, \end{aligned}$$

from which x_1, x_2, x_3 can be found by back substitution.

Note 1 : 1. The factorisation $A = LU$ is unique (if it exists).

2. All principal minors of $A \neq 0$.

Computation of L & U :

Expanding (2), i.e., $A = LU$, we can find the constants, $u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}$ and l_{21}, l_{31}, l_{32}, as follows

$$LU = A \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

First, we get the 1st row of U as,

$$u_{11} = a_{11}, \; u_{12} = a_{12}, \; u_{13} = a_{13}$$

Again, $l_{21}u_{11} = a_{21}$, $l_{31}u_{11} = a_{31}$,

$$\Rightarrow \left.\begin{aligned} l_{21} &= \frac{a_{21}}{u_{11}} = \frac{a_{21}}{a_{11}} \\ l_{31} &= \frac{a_{31}}{u_{11}} = \frac{a_{31}}{a_{11}} \end{aligned}\right\} \text{1}^{\text{st}} \text{ column of L is known}$$

Further, $l_{21}u_{12} + u_{22} = a_{22}$

$l_{21}u_{13} + u_{23} = a_{23}$

Substitution of u_{12}, u_{13}, l_{21} gives,

$$u_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

and $$u_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13} \quad \Bigg\} \text{ 2}^{\text{nd}} \text{ row of U is found}$$

Again, $l_{31}u_{12} + l_{32}u_{22} = a_{32}$

Substitution of l_{31}, u_{12}, and u_{22} we get,

$$l_{32} = \frac{a_{32} - \left(\frac{a_{31}}{a_{11}}\right)(a_{12})}{u_{22}} \quad (\text{2}^{\text{nd}} \text{ column of L is known})$$

Finally, $l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$

from which u_{33} can be found since all other elements are known.

1.4.9 LU Decomposition from Guass Elimination

Consider the system,

$$AX = B \quad \text{..... (1)}$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Using Gauss Elimination Method (1.4.7), the system can be reduced to

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a'_{22}y + a'_{23}z &= b'_3 \\ a''_{33}z &= b''_3 \end{aligned}\right\} \quad \text{..... (2)}$$

The upper triangular matrix (coefficient matrix of 2) is,

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \text{..... (3)}$$

(of the LU Decomposition (verify)).

Again, to reduce the system (1) to (2), we have used elimination method in which,

1. We multiplied 1st equation of (1) by a_{21}/a_{11} and subtracted it from 2nd equation of (1) [This eliminated x from 2nd equation].

2. Multiplied 1st equation of (1) by a_{31}/a_{11} and subtracted from 3rd equation of (1) [This eliminated x from 3rd equation].

In the next step, to eliminate y from 3rd equation, we used the multiplier $\frac{a'_{32}}{a'_{22}}$. If we denote the above 3 multipliers $\frac{a_{21}}{a_{11}}$, $\frac{a_{31}}{a_{11}}$ and $\frac{a'_{32}}{a'_{22}}$, respectively by l_{21}, l_{31}, l_{32}, we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Note: In the computer program, we can store these elements l_{21}, l_{31}, l_{32} of L in places of zeroes of U in (3) as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ l_{21} & a'_{22} & a'_{23} \\ l_{31} & l_{32} & a''_{33} \end{bmatrix};$$

which represents the storage of LU decomposition of A. Clearly, A = LU, where,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}; \; U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \quad \text{(verify)}$$

1.4.10 Tri-diagonal Matrix

Matrices of the type,

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$$ are called tri-diagonal matrices.

1.4.11 Solution of Tri-diagonal Systems

Let $A = \begin{bmatrix} b_1 & c_1 & 0 & 0 & \dots & \dots & 0 \\ a_2 & b_2 & c_2 & 0 & \dots & \dots & 0 \\ 0 & a_3 & b_3 & c_3 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & \dots & 0 & a_n & b_n \end{bmatrix}$

be a tri-diagonal matrix.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $D = \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_n \end{bmatrix}$

then the system of equations,

$$AX = D \qquad \text{..... (1)}$$

in n unknowns $x_1, x_2, \dots, x_n$, is known as a tridiagonal system. The system (1) can be solved by LU decomposition method.

However, a computational method (useful for use on a digital computer) to solve (1) is given below. [This method is due to Thomas].

Method :

Step 1 : Let $\alpha_1 = b_1$, and compute the values of α_i (i = 2, 3, ..., n) using

$$\alpha_i = b_i - \frac{a_i c_{i-1}}{\alpha_{i-1}}, \quad i = 2, 3, \dots, n \qquad \text{..... (1)}$$

Step 2 : Let $\beta_1 = \frac{d_1}{b_1}$ and compute the values of β_i (i = 2, 3, ..., n), using

$$\beta_i = \frac{d_i - a_i \beta_{i-1}}{\alpha_i}, \quad i = 2, 3, \dots, n \qquad \text{..... (2)}$$

Step 3 : Let $x_n = \beta_n$ and compute the values x_i (i = n–1, n–2, ..., 1) using

$$x_i = \beta_i - \frac{c_i x_{i+1}}{\alpha_i}, \quad (i = n-1, n-2, \dots, 1) \qquad \text{..... (3)}$$

1.4.12 Homogeneous Linear equations

The system of equations AX = 0 is said to be homogeneous where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad \text{and } 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$$

Here the augmented matrix is

(A/B) = (A/0)

Hence ρ(A/B) = ρ(A)

Conclusions :

∴ ***(1) The system AX = 0 is always consistent. [Obviously, the trivial solution*** $x_1 = x_2 = \ldots.. = x_n = 0$ ***always exists].***

(2) If ρ(A/B) = ρ(A) = n (i.e., is |A| ≠ 0) then the trival solution is the unique solution.

(3) If ρ(A/B) = ρ(A) = r < n, the system has non-trivial solutions. (In this case |A| = 0).

Let ρ(A/B) = ρ(A) = r < n. Then the system will have (n – r) linearly independent solutions. The values of 'r' unknowns can be expressed in these arbitrarily chosen (n – r) unknowns.

Solved Examples

EXAMPLE 26

Test the system of equations

$$2x + y + 5z = 4;\ 3x - 2y + 2z = 2\ ;\ 5x - 8y - 4z = 1$$

for consistency. If consistent solve them.

SOLUTION

The given system of equations can be put in the matrix form AX = B where,

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & -2 & 2 \\ 5 & -8 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

Augmented matrix

$$(A/B) = \begin{bmatrix} 2 & 1 & 5 & 4 \\ 3 & -2 & 2 & 2 \\ 5 & -8 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} & 2 \\ 3 & -2 & 2 & 2 \\ 5 & -8 & -4 & 1 \end{bmatrix}, R_1 \to \frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} & 2 \\ 0 & \frac{-7}{2} & \frac{-11}{2} & -4 \\ 0 & \frac{-21}{2} & \frac{-33}{2} & -9 \end{bmatrix}, \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} & 2 \\ 0 & 1 & \frac{11}{7} & \frac{8}{7} \\ 0 & \frac{-21}{2} & \frac{-33}{2} & -9 \end{bmatrix} R_2 \to \frac{-2}{7} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{12}{7} & \frac{10}{7} \\ 0 & 1 & \frac{11}{7} & \frac{8}{7} \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{1}{2} R_2 \\ R_3 \to R_3 + \frac{21}{2} R_2 \end{matrix}$$

From the above equivalent matrix, we find that rank of $A = 2$; rank of $(A/B) = 3$

$\therefore \quad \rho(A) \neq \rho(A/B)$

Hence the system is not consistent

$\therefore$ It has no solution.

EXAMPLE 27

Solve the system $2x - y + 4z = 12$; $3x + 2y + z = 10$; $x + y + z = 6$; if it is consistent.

SOLUTION

The matrix form of the system is

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 6 \end{bmatrix} \qquad .....(1)$$

or $\quad AX = B$

$$\text{where} \quad A = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 12 \\ 10 \\ 6 \end{bmatrix}$$

Augmented matrix

$$(A, B) = \begin{bmatrix} 2 & -1 & 4 & 12 \\ 3 & 2 & 1 & 10 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 2 & 1 & 10 \\ 2 & -1 & 4 & 12 \end{bmatrix}, R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -2 & -8 \\ 0 & -3 & 2 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & -3 & 2 & 0 \end{bmatrix} R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 8 & 24 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 3R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} R_3 \to \frac{1}{8} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 2R_3 \end{matrix} \qquad .....(2)$$

which is in Echelon form.

$\rho(A, B) = \rho(A) = 3$

So the system is consistent and has unique solution.

Further (2) is equivalent to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore \quad x = 1 ; y = 2 ; z = 3$

EXAMPLE 28

If consistent, solve the system of equations,

$x + y + z + t = 4$

$x - z + 2t = 2$

$y + z - 3t = -1$

$x + 2y - z + t = 3$

SOLUTION

The system can be written as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 1 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 3 \end{bmatrix} \qquad .....(5)$$

The augmented matrix

$$(A, B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 1 & 2 & -1 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & 1 & -2 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 1 & -2 & 0 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 1 & -2 & 0 & -1 \end{bmatrix} R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & -4 & 1 & -3 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & 1 & -3 \end{bmatrix} R_3 \to -R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & 1 & 0 & -5 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 9 & 9 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 2R_3 \\ R_4 \to R_4 + 4R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & 1 & 0 & -5 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} R_4 \to \frac{1}{9} R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 4R_4 \\ R_2 \to R_2 + 5R_4 \\ R_3 \to R_3 - 2R_4 \end{matrix} \quad .....(2)$$

which is in Echelon form

ρ(A) = ρ(A/B) = 4 = number of unknowns

∴ The system (1) is consistent and has a unique solution.

Again from the reduced form (2) of (A/B), the system becomes,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

⇒ $x = 1,\ y = 1,$

$z = 1$, and $t = 1$;

EXAMPLE 29

Test for consistency and if consistent solve the system,

$$5x + 3y + 7t = 4$$
$$3x + 26y + 2t = 9$$
$$7x + 2y + 10t = 5$$

SOLUTION

The given system can be written as

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} \quad .....(1)$$

or $AX = B,$

where $A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

The augmented matrix

$$(A,B) = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 2 & -1 & 3 & 1 \end{bmatrix} R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 1 & 27 & -1 & 8 \\ 2 & -1 & 3 & 1 \end{bmatrix} R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 27 & -1 & 8 \\ 5 & 3 & 7 & 4 \\ 2 & -1 & 3 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 27 & -1 & 8 \\ 0 & -132 & 12 & -36 \\ 0 & -55 & 5 & -15 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 27 & -1 & 8 \\ 0 & 1 & -\frac{1}{11} & \frac{3}{11} \\ 0 & -55 & 5 & -15 \end{bmatrix} R_2 \to \frac{-1}{132} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 16/11 & 7/11 \\ 0 & 1 & -1/11 & 3/11 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 27R_2 \\ R_3 \to R_3 + 55R_2 \end{matrix} \quad \text{.....(2)}$$

which is in Echelon form.

$\therefore$ $\rho(A, B) = \rho(A) = 2 <$ the number of unknowns.

$\therefore$ The system (1) is consistent with infinite number of solutions.

$\rho(A) = r = 2$, number of unknowns n =3.

Hence we have to consider (3 – 2 = 1) one variable as an arbitrary constant and the other two variables are expressed in terms of this constant.

The final reduced system (2) can be written as

$$\left.\begin{aligned} x+\frac{16t}{11}=\frac{7}{11} \\ y-\frac{t}{11}=\frac{3}{11} \end{aligned}\right\} \quad .....(3)$$

$$\Rightarrow \qquad y=\frac{3+t}{11}\,;\, x=\frac{7-16t}{11}$$

If $t = c$, where c is an arbitrary constant, we have, the solution of the system (1) as,

$$x=\frac{1}{11}(7\text{ -}16c),\, y=\frac{1}{11}\,(c+3),\, t=c$$

For different values of 'c' we get different solutions. Thus we get infinite number of solutions.

EXAMPLE 30

Solve the system $x - z - 2 = 0$; $y - z + 2 = 0$; $x + y - 2z = 0$; if it is consistent.

SOLUTION

The system is $AX = B$, where

$$A=\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix},\, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix},\, B=\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

Augmented matrix

$$(A,B)=\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & -2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{bmatrix} R_3 \to R_3 - R_1$$

$$\sim\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

$\rho(A, B) = \rho(A) = 2 <$ number of unknowns 3.

So the system being consistent, will have infinite number of solutions which involve $3 - 2 = 1$ arbitrary constant.

The final reduced matrix $\sim (A,B)$ makes the system equivalent to,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$\Rightarrow \quad x - z = 2, y - z = -2$

$\Rightarrow \quad x = z + 2, y = z - 2$

Taking $z = c$ (an arbitrary constant) we get the solution as,

$x = c + 2, y = c - 2, z = c.$

EXAMPLE 31

Investigate for what values of α, β, the system of equations given by

(JNTU 2001, 2002, 2004, 2005)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \alpha \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \beta \end{bmatrix}, \text{ has}$$

(i) no solution (ii) unique solution (iii) an infinite number of solutions.

SOLUTION

The given system can be put in the form

$$AX = B, \text{ where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \alpha \end{bmatrix}, \text{X} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \text{B} = \begin{bmatrix} 6 \\ 10 \\ \beta \end{bmatrix}$$

Augmented matrix

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \alpha & \beta \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & (\alpha - 1) & (\beta - 6) \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & (\alpha-3) & (\beta-10) \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$= C$ (say)

Case (i) :

If the system has no solution,

$\rho(A/B) \neq \rho(A)$

i.e., $\alpha = 3$, $\beta \neq 10$

Explanation :

If $\alpha = 3$; $\beta \neq 10$;

$$A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \rho(A) = 2 \text{ and } (A, B) \sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & \gamma \end{bmatrix}$$

(where $\gamma = \beta - 10 \neq 0$)

$$\therefore \quad \begin{vmatrix} 0 & -1 & 2 \\ 1 & 2 & 4 \\ 0 & 0 & \gamma \end{vmatrix} \neq 0, \text{ so } \rho(A,B) = 3$$

Hence $\rho(A) \neq \rho(A, B)$

$\therefore$ System has no solution.

Case (ii) :

If the system has a unique solution, $\rho(A) = \rho(A,B)$ = number of unknowns = 3

$\therefore$ $\alpha - 3 \neq 0$, and $(\beta - 10)$ can take any value

i.e., $\alpha \neq 3$, and β can have any value

Explanation : $\alpha \neq 3 \Rightarrow C = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \delta & \beta-10 \end{bmatrix}$ (where $\delta = \alpha - 3 \neq 0$)

$$\therefore \quad \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & \delta \end{vmatrix} = \delta \neq 0$$

$\therefore$ $\rho(A) = \rho(A, B) = 3$

Case (iii) :

The system has infinite number of solutions (say).

Then $\rho(A) = \rho(A, B) < 3$ number of unknowns.

$$\therefore \text{ If } \alpha = 3, \text{ and } \beta = 10, C = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so that } \rho(A) = \rho(A, B) = 2 < 3$$

$\therefore \quad \alpha = 3$ and $\beta = 10$.

Hence the required answers are

(i) $\alpha = 3, \beta \neq 10$

(ii) $\alpha \neq 3$, β can have any value

(iii) $\alpha = 3$ and $\beta = 10$.

EXAMPLE 32

Test the following system for consistency and if consistent solve it.

$$u + 2v + 2w = 1,\ 2u + v + w = 2,\ 3u + 2v + 2w = 3,\ v + w = 0.$$

SOLUTION

The system is in the form $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

Augmented matrix

$$(A,B) = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 \to \frac{-1}{3} R_2 \\ R_3 \to \frac{-1}{4} R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

Which is in Echelon form with two nonzero rows.

$\therefore \quad \rho(A) = \rho(A, B) = 2 < 3$ (number of unknowns)

$\therefore$ The system is consistent and has infinite solutions containing $3 - 2 = 1$ arbitrary constant 'c' say.

The system is reduced to

$$\left.\begin{matrix} u + 2v + 2w = 1 \\ v + w = 0 \end{matrix}\right\} \text{[from final equivalent matrix]}$$

Let $\quad w = c$, then $v = -c$

and $\quad u = 1$ ($\because 2v + 2w = 0$)

$$\therefore \quad X = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -c \\ c \end{bmatrix},$$

where c is arbitrary, is the solution.

EXAMPLE 33

Solve the following system completely.

$x + y + z = 1$; $x + 2y + 4z = \alpha$

$x + 4y + 10z = \alpha^2$

SOLUTION

The system can be put in the matrix form $AX = B$, as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix} \qquad .....(1)$$

Here $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}$

Augmented matrix

$$(A,B) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \alpha \\ 1 & 4 & 10 & a^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \alpha-1 \\ 0 & 3 & 9 & \alpha^2-1 \end{bmatrix} R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 2-\alpha \\ 0 & 1 & 3 & \alpha-1 \\ 0 & 0 & 0 & \alpha^2-3\alpha+2 \end{bmatrix} R_1 \to R_1 - R_2,\ R_3 \to R_3 - 3R_2$$

$= C(\text{say})$

For the system to be consistent, $\rho(A)$ and $\rho(A, B)$ must be equal.

$\because \quad A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \rho(A) = 2$

$\therefore \quad \rho(A, B) = \rho(C) = 2 \Rightarrow \alpha^2 - 3\alpha + 2 = 0 \Rightarrow \alpha = 1$ or $\alpha = 2$.

$\therefore$ The system is consistent when $\alpha = 1$ or 2.

Case (i) :

$$\alpha = 1 \Rightarrow \text{C} = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad x - 2z = 1, y + 3z = 0$ .....(2)

$\because$ ρ(A) = ρ(A, B) = 2 < number of unknowns 3, the system has infinite solutions involving one arbitrary constant.

$\therefore$ Let $z = c_1$, so that (2) gives $y = -3z$

$\Rightarrow \quad y = -3\,c_1$ and $x = 1 + 2\,z$

$\Rightarrow \quad x = 1 + 2c_1$

$\therefore$ when $\alpha = 1$, the solution is,

$x = 1 + 2c_1,$
$y = -3c_1,$
$z = c_1,$

where c_1 is an arbitrary constant.

Case (ii) :

$$\alpha = 2 \Rightarrow \mathrm{C} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad x - 2z = 0, y + 3z = 1,$

Taking $z = c_2$ an arbitrary constant, we get the solution as,

$x = 2c_2,$
$y = 1 - 3c_2,$
$z = c_2$

Example 34

Show that the system

$$3x + 4y + 5z = \alpha,$$
$$4x + 5y + 6z = \beta,$$
$$5x + 6y + 7z = \gamma,$$

is consistent only when α, β, γ are in A . P.

Solution

The given system, when put in matrix form is

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Which when compared with $AX = B$ gives

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Augmented matrix

$$(A, B) = \begin{bmatrix} 3 & 4 & 5 & \alpha \\ 4 & 5 & 6 & \beta \\ 5 & 6 & 7 & \gamma \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \beta-\alpha \\ 1 & 1 & 1 & \gamma-\beta \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \beta-\alpha \\ 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \gamma-\beta \end{bmatrix}, R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \beta-\alpha \\ 0 & 1 & 2 & 4\alpha-3\beta \\ 0 & 0 & 0 & \gamma-2\beta+\alpha \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$\Rightarrow$ $\rho(A) = 2$ for any α, β, γ, and $\rho(A, B) = 2$ when $\alpha - 2\beta + \gamma = 0$.

But the system is consistent if and only if

$\rho(A) = \rho(A, B)$

i.e., if $\rho(A, B) = 2$,

i.e., if $\alpha - 2\beta + \gamma = 0$

i.e., if $\alpha + \gamma = 2\beta$,

i.e, when α, β, γ are in A.P.

EXAMPLE 35

Find the totality of solutions of the system given by,

$$5x + 2y - 6z + 2t + 1 = 0$$

$$x - y + z - t + 2 = 0$$

SOLUTION

The given system is,

$$\begin{bmatrix} 5 & 2 & -6 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Which is in the form $AX = B$, where

$$A = \begin{bmatrix} 5 & 2 & -6 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}, B = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Augmented matrix D,

$$(A, B) = \begin{bmatrix} 5 & 2 & -6 & 2 & -1 \\ 1 & -1 & 1 & -1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 2 & -6 & 2 & -1 \\ 0 & \frac{-7}{5} & \frac{11}{5} & \frac{-7}{5} & \frac{-9}{5} \end{bmatrix} \left(R_2 \to R_2 - \frac{1}{5} R_1 \right)$$

$\because$ $\begin{vmatrix} 5 & 2 \\ 0 & \frac{-7}{5} \end{vmatrix} \neq 0$ is a minor of order 2 of A and

$\begin{vmatrix} 5 & -1 \\ 0 & \frac{-9}{5} \end{vmatrix} \neq 0$ is a minor of order 2 of (A, B) .

$\rho(A) = \rho(A, B) = 2 <$ number of unknowns,

$\therefore$ The system is consistent and has infinite solutions which involve $4 - 2 = 2$ arbitrary constants.

The final - reduced matrix makes the system as,

$$5x + 2y - 6z + 2t = -1 \qquad \text{.....(1)}$$

and $$\frac{-7}{5} y + \frac{11}{5} z \frac{-7}{5} t = \frac{-9}{5}$$

(or) $$-7y + 11z - 7t = -9 \qquad \text{.....(2)}$$

Taking $z = k_1$, $t = k_2$, (k_1, k_2 being arbitrary constants)

we get, from (2),

$$y = \frac{1}{7} (11 k_1 - 7k_2 + 9)$$

$$(1) \Rightarrow \quad x = \frac{1}{5}\,(-2y + 6z - 2t - 1)$$

$$= \frac{1}{5}\left[\frac{-2}{7}(11k_1 - 7k_2 + 9) + 6k_1 - 2k_2 - 1\right]$$

$$= \frac{1}{7}\,(4k_1 - 5).$$

$\therefore$ The totality of solutions of the system is

$$x = \frac{1}{7}(4k_1 - 5),\ y = \frac{1}{7}(11k_1 - 7k_2 + 9),\ z = k_1,\ t = k_2$$

where k_1, k_2 are arbitrary constants.

EXAMPLE 36

Solve the system

$$x + y + z = 6$$
$$2x - 3y + 4z = 8$$
$$x - y + 2z = 5, \text{ by the methods}$$

(i) Matrix inversion method

(ii) Using cramer's rule

(iii) Gauss Jordan method

SOLUTION

The system is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 5 \end{bmatrix}$$

If it is put in the form $AX = B$, we have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 1 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\ \mathrm{B} = \begin{bmatrix} 6 \\ 8 \\ 5 \end{bmatrix}$$

(i) Matrix inversion method :

We find A^{-1}

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\text{Adjoint of} \quad A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Where A_{11}, A_{12}, ... are cofactors of a_{11}, a_{12}, ... respectively.

$$A_{11} = (-1)^{1+1} \begin{vmatrix} -3 & 4 \\ -1 & 2 \end{vmatrix} = -2 \; ; A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 0;$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} = +1;$$

$$|A| = \det A = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$
$$= 1.(-2) + 1(0) + 1\,(1) = -1$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3,$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 1 \; ; A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2$$

$$A_{31} = (-1)^{1+3} \begin{vmatrix} 1 & 1 \\ -3 & 4 \end{vmatrix} = 7 \; ; A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -2 \; ;$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$$

$$\therefore \quad \text{Adj}\, A = \begin{bmatrix} -2 & -3 & 7 \\ 0 & 1 & -2 \\ 1 & 2 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj}\, A = -\begin{bmatrix} -2 & -3 & 7 \\ 0 & 1 & -2 \\ 1 & 2 & -5 \end{bmatrix}$$

Solution matrix

$$X = A^{-1} B$$

$$\therefore \quad X = \frac{+1}{-1}\begin{bmatrix} -2 & -3 & 7 \\ 0 & 1 & -2 \\ 1 & 2 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 5 \end{bmatrix}$$

$$= -\begin{bmatrix} -2.6 & -3.8 & +7.5 \\ 0.6 & +1.8 & -2.5 \\ 1.6 & +2.8 & -5.5 \end{bmatrix}$$

$$= -\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ $x = 1, y = 2, z = 3$ is the solution.

(ii) Cramer's Rule :

$$\Delta = |A| = -1$$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 8 & -3 & 4 \\ 5 & -1 & 2 \end{vmatrix} = 6(-6+4) - 1(16-20) + 1(-8+15)$$

$$= -12 + 4 + 7 = -1$$ Replacing 1st column of Δ by B

$$\text{Similarly, } \Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 8 & 4 \\ 1 & 5 & 2 \end{vmatrix} = 1(16-20) - 6(4-4) + 1(10-8)$$

$$= -4 - 0 + 2 = -2$$

$$\text{and} \quad \Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 2 & -3 & 8 \\ 1 & -1 & 5 \end{vmatrix} = 1(-15+8) - 1(10-8) + 6(-2+3)$$

$$= -7 - 2 + 6 = -3$$

$\therefore$ solution is

$$x = \frac{\Delta_1}{\Delta} = 1; \; y = \frac{\Delta_2}{\Delta} = 2; \; z = \frac{\Delta_3}{\Delta} = 3$$

(iii) Gauss-Jordan method :

$$[A,\ B]=\begin{bmatrix}1 & 1 & 1 & 6\\ 2 & -3 & 4 & 8\\ 1 & -1 & 2 & 5\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & -5 & 2 & -4\\ 0 & -2 & 1 & -1\end{bmatrix}\begin{matrix}R_2\to R_2-2R_1\\ R_3\to R_3-R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 1 & -2/5 & \frac{4}{5}\\ 0 & -2 & 1 & -1\end{bmatrix}\sim R_2\to -\frac{1}{5}R_2$$

$$\sim\begin{bmatrix}1 & 0 & 7/5 & 26/5\\ 0 & 1 & -2/5 & 4/5\\ 0 & 0 & 1/5 & 3/5\end{bmatrix}\begin{matrix}R_1\to R_1-R_2\\ R_3\to R_3+2R_2\end{matrix}$$

$$\sim\begin{bmatrix}1 & 0 & 7/5 & 26/5\\ 0 & 1 & -2/5 & 4/5\\ 0 & 0 & 1 & 3\end{bmatrix}R_3\to 5R_3$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 3\end{bmatrix}\begin{matrix}R_1\to R_1-\frac{7}{5}R_3\\ R_2\to R_2+\frac{2}{5}R_3\end{matrix}$$

$$=[I_3\ X]$$

where $X=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$

∴ solution is

$x=1, y=2, z=3$

EXAMPLE 37

Show that the system

$$2x - 3y + z = 0,$$
$$4x + 9y + z = 0,$$
$$8x - 27y + z = 0,$$ has no non-trivial solution.

SOLUTION

The given system can be written as

$$\begin{bmatrix} 2 & -3 & 1 \\ 4 & 9 & 1 \\ 8 & -27 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which is of the form

$$AX = 0,$$

where $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 9 & 1 \\ 8 & -27 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 9 & 1 \\ 8 & -27 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3/2 & 1/2 \\ 4 & 9 & 1 \\ 8 & -27 & 1 \end{bmatrix} R_1 \to \frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 15 & -1 \\ 0 & -15 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 8R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & -1/15 \\ 0 & -15 & -3 \end{bmatrix} R_2 \to \frac{1}{15} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & -1/15 \\ 0 & 0 & -4 \end{bmatrix} \begin{matrix} R_1 \to R_1 + \frac{3}{2} R_2 \\ R_3 \to R_3 + 15R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & -1/15 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to \frac{-1}{4} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2/5\, R_3 \\ R_2 \to R_2 + 1/15\, R_3 \end{matrix}$$

Which is in the canonical form I_3.

$\therefore$ $\rho(A) = 3$ = number of unknowns.

$\therefore$ The system is consistent and has unique solution which is the trivial solution $x = 0$, $y = 0$, $z = 0$.

(ALITER : Since A is a 3×3 matrix, it is easy to find $\rho(A)$ by showing that $|A| \neq 0$.)

EXAMPLE 38

Solve the system of homogeneous equations given by

$2x + y + 2z = 0$,

$x + y + 3z = 0$,

$4x + 3y + 8z = 0$.

SOLUTION

The given system is of the form $AX = 0$ where

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & 8 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 8 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix} \begin{matrix} R_2 \to -R_2 \\ R_3 \to -R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2,$$

$\therefore \quad \rho(A) = 2,$

Hence the system is consistent and has infinite solutions with one arbitrary constant.

The final reduced matrix reduces the system to

$$x + y + 3z = 0$$

$$y + 4z = 0$$

Taking $z = k$, we get $y = -4k$, and $x = k$ (k being arbitrary constant)

$\therefore$ The solution of the system is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -4k \\ k \end{bmatrix}$$

EXAMPLE 39

Solve the system,

$$x + y = 0,$$

$$x + 2y + 3z = 0,$$

$$u + v + w = 0,$$

$$x + u + w = 0.$$

SOLUTION

In this problem, we have to find 6 variables x, y, z, u, v, w.

The given system, when written as $AX = 0$, is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

we find that, $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}$

$= 3 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$, is the largest non- vanishing minor of A.

$\therefore \quad \rho(A) = 4$

Hence the system has infinite solutions and involves $6 - 4 = 2$ arbitrary constants.

Taking $v = k_1, w = k_2$ (k_1, k_2 being arbitrary constants)

We have, $u = -k_1 - k_2$

$x = k_1, y = -k_1$

$z = \frac{1}{3} k_1$

$\therefore$ The solution is

$$x = +k_1,$$
$$y = -k_1,$$
$$z = \frac{1}{3}k_1,$$
$$u = -k_1 - k_2,$$
$$v = k_1,$$
$$w = k_2$$

Note : $\rho(A)$ can also be found by any alternate method.

EXAMPLE 40

If $a, b, c,$ are distinct non-zero numbers, show that the homogeneous system

$$\begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has no non-trivial solution.

SOLUTION

Here $AX = 0$ is the system where

$$A = \begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$ (Taking a, b, c common from the three columns).

$$= abc \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \quad \begin{matrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{matrix}$$

$$= abc\,.\,(b-a)\,(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix} \quad \begin{matrix} c_2 \to \dfrac{1}{b-a}c_2 \\ c_3 \to \dfrac{1}{c-a}c_3 \end{matrix}$$

$$= abc\,(b-a)\,(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$$

$= abc\,(b-a)\,(c-a)\,(c-b) \neq 0$, since a, b, c are distinct non-zero numbers.

$\therefore\ \rho(A) = 3$ = number of unknowns. Hence the system has a unique trivial solution $x = 0, y = 0, z = 0$.

Hence the problem ($\rho(A)$ can also be found by another method)

EXAMPLE 41

Apply Gauss Elimination and solve the system

$$2x + 3y + 4z = 9 \quad \text{.....(1)}$$

$$3x + y + 2z = 6 \quad \text{.....(2)}$$

$$x + y + 3z = 5 \quad \text{.....(3)}$$

SOLUTION

Step 1 : Elimination

$$(1) \times \frac{-3}{2} + (2) \Rightarrow \frac{-7}{2}y - 4z = \frac{-15}{2} \Rightarrow 7y + 8z = 15 \quad \text{.....(4)}$$

$$(1) \times -½ + (3) \Rightarrow \frac{-1}{2}y + z = ½ \Rightarrow -y + 2z = 1 \quad \text{.....(5)}$$

$$(4) \times \frac{1}{7} + (5) \Rightarrow \frac{22}{7}z = \frac{22}{7} \Rightarrow 22z = 22 \Rightarrow z = 1 \quad \text{.....(6)}$$

Thus, (1), (4) and (6) form an upper triangular system.

Step 2 : Back Substituting z in (4) $\Rightarrow y = 1$

$(1) \Rightarrow x = 1$

$\therefore\ x = 1, y = 1, z = 1$, is the solution

EXAMPLE 42

Solve the system

$$x + 2y + 3z = 10$$

$$3x + y + 2z = 13$$

$$2x + 3y + z = 13 \qquad \text{by LU Decomposition Method}$$

SOLUTION

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} = LU$, where,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

$$\therefore \quad \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

Clearly, $u_{11} = 1, u_{12} = 2, u_{13} = 3$

Again, $l_{21}u_{11} = 3 \Rightarrow l_{21} = 3$

$l_{31}u_{11} = 2 \Rightarrow l_{31} = 2$

Also, $l_{21}u_{12} + u_{22} = 1 \Rightarrow 3 \times 2 + u_{22} = 1 \Rightarrow u_{22} = -5$

$l_{21}u_{13} + u_{23} = 2 \Rightarrow 3 \times 3 + u_{23} = 2 \Rightarrow u_{23} = -7$

Finally, $l_{31}u_{12} + l_{32}u_{22} = 3 \Rightarrow 2 \times 2 + l_{32}(-5) = 3 \Rightarrow l_{32} = 1/5$

and $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1 \Rightarrow 2 \times 3 + (1/5)(-7) + u_{33} = 1$

$$\Rightarrow \quad u_{33} = 1 - 6 + \frac{7}{5} = -\frac{18}{5}$$

$\therefore$ The given system becomes, LUX = B,

$$\therefore \quad \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1/5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 13 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 13 \end{bmatrix} \quad \text{.....(I)}$$

where
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \text{.....(II)}$$

I $\Rightarrow$ $y_1 = 10$

$3y_1 + y_2 = 13 \Rightarrow y_2 = -17$

$2y_1 + (1/5)y_2 + y_3 = 13 \Rightarrow y_3 = 13 - 20 + (17/5) \Rightarrow y_3 = -(18/5)$

II $\Rightarrow$ $x + 2y + 3z = 10$

$-5y - 7z = -17$

$$-\frac{18}{5}z = -\frac{18}{5}$$

From which back substitution gives,

$z = 1, y = 2$ and $x = 3$

EXAMPLE 43

Solve the Tri-diagonal system of equations,

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

SOLUTION

(**Note:** we follow here the method given in 1.4.11)

Here number of unknowns, n = 4 (x, y, z and t)

$$b_1 = b_2 = b_3 = b_4 = 2;$$

$$a_1 = a_3 = a_4 = -1;$$

$$c_1 = c_2 = c_3 = -1;$$

and $$d_1 = d_2 = d_3 = 0;\ d_4 = 1$$

Step 1 : $\alpha_1 = b_1 = 2$

From (1), $\alpha_i = b_i - \dfrac{a_i c_{i-1}}{\alpha_{i-1}}$ i = 2, 3, 4, ...

$$\therefore \quad \alpha_2 = b_2 - \frac{a_2 c_1}{\alpha_1} = 2 - \frac{(-1)(-1)}{2} = \frac{3}{2}$$

$$\alpha_3 = b_3 - \frac{a_3 c_2}{\alpha_2} = 2 - \frac{(-1)(-1)}{3/2} = \frac{4}{3}$$

$$\alpha_4 = b_4 - \frac{a_4 c_3}{\alpha_3} = 2 - \frac{(-1)(-1)}{4/3} = \frac{5}{4}$$

Step 2 : $\beta_1 = \dfrac{d_1}{b_1} = 0$

From (2), $\beta_i = \dfrac{d_i - a_i\beta_{i-1}}{\alpha_i}$, i = 2, 3, 4

$$\therefore \quad \beta_2 = \frac{d_2 - a_2\beta_1}{\alpha_2} = 0 \quad [\text{Since } d_2 = \beta_1 = 0]$$

$$\beta_3 = \frac{d_3 - a_3\beta_2}{\alpha_3} = 0 \quad [\text{Since } d_3 = \beta_2 = 0]$$

$$\beta_4 = \frac{d_4 - a_4\beta_3}{\alpha_4} = \frac{1-0}{5/4} = \frac{4}{5}$$

Step 3 : $x_4 = t = \beta_4 = \frac{4}{5} = 0.8$

From (3), $x_i = \beta_i - \frac{c_i x_{i+1}}{\alpha_i}, i = 3, 2, 1$

$$i = 3 \Rightarrow \quad z = x_3 = \beta_3 - \frac{c_3 x_4}{\alpha_3} = 0 - \frac{(-1)\left(\frac{4}{5}\right)}{4/3} = \frac{3}{5}$$

$$i = 2 \Rightarrow \quad y = x_2 = \beta_2 - \frac{c_2 x_3}{\alpha_2} = 0 - \frac{(-1)\left(\frac{3}{5}\right)}{3/2} = \frac{2}{5}$$

$$i = 1 \Rightarrow \quad x = x_1 = \beta_1 - \frac{c_1 x_2}{\alpha_1} = 0 - \frac{(-1)\left(\frac{2}{5}\right)}{2} = \frac{1}{5}$$

Hence, $x = 0.2, y = 0.4, z = 0.6$ and $t = 0.8$

Exercise 1(e)

I **Test for consistency the following systems. If consistent, solve them.**

(i) $x - 2y + t = 4$; $3x + 5y + t = 6, 6x - y + 4t = 2$

(Ans : not consistent)

(ii) $4x - y + 3z = 11, 2x + y - 3z = -5, x + y + z = 6$

(Ans : $x = 1, y = 2, z = 3$)

(iii) $x - y + z = 3, 2x - 3y + 5z = 10, x + y + 4z = 4$

(Ans : $x = 1, y = -1, z = 1$)

(iv) $x_1 + x_2 - x_3 + x_4 = 2$; $3x_1 - x_2 + 2x_3 + 5x_4 = 9$
$4x_1 + x_2 - x_3 - x_4 = 3$; $2x_1 - x_2 + 3x_3 + x_4 = 5$

(Ans : $x_1 = x_2 = x_3 = x_4 = 1$)

(v) $x + y + 2z + t = 5, 2x + 3y - z - 2t = 2, 4x + 5y + 3z = 7$

(Ans : Inconsistent)

(vi) $u + v + w = 6,\ u + 2v + 3w = 14;\ u + 4v + 7w = 30$

(Ans : $u = k - 2,\ v = 8 - 2k,\ w = k$, k is arbitrary constant)

(vii) $u + v + w + t = 1;\ v + w = 3,\ u - t + 2 = 0,\ 2u + 3v + 3w - 2t = 5$

(Ans : $u = k_1 - 2,\ v = 3 - k_2,\ w = k_2,\ t = k_1$, k_1, k_2 being arbitrary constants)

(viii) $x + y + z = 6,\ x - y + 2z = 5,\ 3x + y + z = 8,\ 2x - 2y + 3z = 7$

(Ans : $x = 1, y = 2, z = 3$)

(ix) $x + 2y + z = 8,\ 2x + 3y + z = 13,\ x + y = 5$

(Ans : $x = 2 + k, y = 3 - k, z = k$)

II Test for consistency the following homogeneous equations. If consistent, solve them :

(i) $x + y = 0, y + z = 0,\ z + x = 0$

(Ans : $x = y = z = 0$)

(ii) $x - 2y + 3z = 0,\ 2x + 5y + 6z = 0,\ 4x + y + 12z = 0$

(Ans : $x = -3k,\ y = 0,\ z = k$, k is constant.)

(iii) $u + v + w + t = 0,\ u + v + w - t = 0$
$u + v - w + t = 0,\ u - v + w + t = 0$

(Ans : $u = v = w = t = 0$)

(iv) $3x_1 + 4x_2 - x_3 - 6x_4 = 0,\ 2x_1 + 3x_2 + 2x_3 - 3x_4 = 0$
$2x_1 + x_2 - 14x_3 - 9x_4 = 0,\ x_1 + 3x_2 + 13x_3 + 3x_4 = 0$

(Ans : $x_1 = 11c_1 + 6c_2,\ x_2 = -8c_1 - 3c_2,\ x_3 = c_1,\ x_4 = c_2$,
when c_1, c_2 are arbitrary constants)

(v) $$\begin{bmatrix} 2 & -3 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Ans : $x = 3k,\ y = 2k,\ z = -k,\ t = k$, k being an arbitrary constant)

III Find the value (s) of 'a' if the systems, $2x - y + 3z = 2,\ x + y + 2z = 2,\ 5x - y + az = 6$ has a unique solution. Find the solution when $a = 10$.

(Ans : $a \neq 8,\ x = 4/3,\ y = 2/3,\ z = 0$)

IV Determine the values of 'a' and 'b' so that the system of equations

$$x + 2y + 3z = 6,\ x + 3y + 5z = 9,\ 2x + 5y + az = b$$

has (i) unique solution (ii) no solution (iii) infinite number of solutions

(Ans : (i) $a \neq 8$, μ any value (ii) $a = 8$, $b \neq 15$ (iii) $a = 8$, $b = 15$**)**

V Find the value (s) of λ such that the system,

$$2x + y + 2z = 0,\ x + y + 3z = 0,\ 4x + 3y + \lambda z = 0$$

has a non-trivial solution.

(Ans : $\lambda \neq 8$**)**

VI Find the value (s) of 'α' for which the system of equations,

$$(2 - \alpha)u - 2v + w = 0$$
$$2u - (3 + \alpha)v + 2w = 0$$
$$u - 2v + \alpha w = 0$$

has a non-trivial solution. Solve the system for such value (s) of 'α'.

(Ans : 1. $\alpha = 1$; $u = 2c_1 - c_2$, $v = c_1$, $w = c_2$
2. $\alpha = -3$; $u = -c$, $v = -2c$, $w = c$,
where c, c_1 c_2 are arbitrary constants**)**

VII Determine the value (s) of 'k' such that the following system may possess a non- trival solution. Also determine the general solution for each permissible value of k.

$$\begin{bmatrix} 3 & 1 & -k \\ 4 & -2 & -3 \\ 2k & 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Ans : 1. $k = 1$; $x = c$, $y = -c$, $z = 2c$
2. $k = -9$; $x = 3c$, $y = 9c$, $z = -2c$
where c is an arbitrary constant.**)**

VIII Solve the systems

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \\ 16 \end{bmatrix}$

by (i) Matrix inversion method.
(ii) Cramer's rule
(iii) Gauss-Jordan method.
(iv) LU Decomposition method.
(v) Gauss Elimination method.

(Ans : (i) $x = 1, y = 1, z = 1$
(ii) $x = 2, y = 3, z = 4$**)**

1.5 Gauss-Siedel Iterative Method

Gauss-Siedel method is the most popular method to find the solution of simultaneous linear algebraic equations. This method is also called method of successive displacements.

Let us consider a set of three equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let $|a_1| \leq |b_1| + |c_1|; |b_2| \leq |a_2| + |c_2|; |c_3| \leq |a_3| + |b_3|$; i.e., in each equation the coefficients of the diagonal terms are large.

Given equations may be written as $x = \frac{1}{a_1}(d_1 - b_1\, y - c_1z)$ .....(1)

$$y = \frac{1}{b_2}(d_2 - a_2\, x - c_2z) \quad .....(2)$$

$$z = \frac{1}{c_3}(d_3 - a_3\, x - b_3y) \quad .....(3)$$

Here start with the initial approximations y_0, z_0 for y, z respectively.

Substituting $y = y_0, z = z_0$ in (1), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1\, y_0 - c_1z_0)$$

Then putting $x = x_1, z = z_0$ in (2), we get

$$y_1 = \frac{1}{b_2}(d_2 - a_2\, x_1 - c_2z_0)$$

Now putting $x = x_1, y = y_1$ in (3), we get

$$z_1 = \frac{1}{c_3}(d_3 - a_3\, x_1 - b_3y_1)$$

By taking x_1, y_1, z_1 as initial approximations for x, y and z using equations (1), (2), and (3), the next approximations for x, y, z are given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1\, y_1 - c_1z_1)$$

$$y_2 = \frac{1}{b_2}(d_3 - a_2\ x_2 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3\ x_2 - b_3 y_2)$$

This procedure is repeated till the values of x, y, z are obtained to desired degrees of accuracy.

This method is very useful with less work for the given system of equations whose augmented matrix have a large number of zero elements.

We say a matrix is diagonally dominant if the numerical value of the leading diagonal element in each row is greater than or equal to the sum of the numerical values of the other elements in that row.

For the Gauss-seidal method to converge quickly the coefficient matrix must be diagonally dominant. If it is not so, we have to rearrange the equations in such a way that the coefficient matrix is diagonally dominant and then only we can apply Gauss-Seidal method.

Solved Examples

EXAMPLE 44

Using Gauss-Seidel iteration method, solve the equations

$$10x - y + 2z = 4$$

$$x + 10y - z = 3$$

$$2x + 3y + 20z = 7$$

SOLUTION

The given equations are written in the form

$$x = \frac{1}{10}(4 + y - 2z) \quad .....(1)$$

$$y = \frac{1}{10}(3 - x + z) \quad .....(2)$$

$$z = \frac{1}{20}(7 - 2x - 3y) \quad .....(3)$$

Start the iteration with an approximation of $y_0 = z_0 = 0$

Substituting $y = y_0, z = z_0$ in (1), we get

$$x_1 = \frac{1}{10}(4 + y_0 - 2z_0) = \frac{1}{10}(4 + 0 - 0) = 0.4$$

Substituting $x = x_1, z = z_0$ in (2), we get

$$y_1 = \frac{1}{10}(3 - x_1 + z_0) = \frac{1}{10}(3 - 0.4 + 0) = 0.26$$

Substituting $x = x_1, y = y_1$ in (3), we get

$$z_1 = \frac{1}{20}(7 - 2x_1 - 3y_1) = \frac{1}{20}[7 - 2(0.4) - 3(0.26)] = 0.271$$

Similarly from the second iteration, we have

$$x_2 = \frac{1}{10}(4 + y_1 - 2z_1) = \frac{1}{10}[4 + 0.26 - 2(0.271)] = 0.3718$$

$$y_2 = \frac{1}{10}(3 - x_2 + z_1) = \frac{1}{10}[3 - 0.3718 + 0.271] = 0.2899$$

$$z_2 = \frac{1}{20}(7 - 2x_2 - 3y_2) = \frac{1}{20}[7 - 2(0.3718) - 3(0.2899)] = 0.2693.$$

Similarly from the Third iteration, we have

$$x_3 = \frac{1}{10}(4 + y_2 - 2z_2) = \frac{1}{10}[4 + 0.2899 - 2(0.2693)] = 0.3751$$

$$y_3 = \frac{1}{10}(3 - x_3 + z_2) = \frac{1}{10}[3 - 0.3751 + 0.2693] = 0.2894$$

$$z_3 = \frac{1}{20}(7 - 2x_3 - 3y_3) = \frac{1}{20}[7 - 2(0.3751) - 3(0.2894)] = 0.2690.$$

Now from third and second iterations, the values of x, y, z are $x = 0.3751$, $y = 0.2894$, $z = 0.2690$, which are correct up to two decimal places. If we want more accurate results, we have to take up few more iterations.

EXAMPLE 45

Solve the system of equations

$$8x - y + z = 18,\ 2x + 5y - 2z = 3,\ x + y - 3z = -6$$

Using Gauss-Seidal iteration method.

SOLUTION

We can express the given system of equations in the form of

$$\begin{pmatrix} 8 & -1 & 1 \\ 2 & 5 & -2 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18 \\ 3 \\ -6 \end{pmatrix}$$

The coefficient matrix is diagonally dominant.

Hence we can apply Gauss-Seidal method

$$x = \frac{1}{8}(18 + y - z) \quad \ldots(1)$$

$$y = \frac{1}{5}(3 + 2x - 2z) \quad \ldots(2)$$

$$z = \frac{1}{3}(6 + x + y). \quad \ldots(3)$$

FIRST ITERATION

Substituting y = 0, z = 0, in (1)

$$x_1 = \frac{1}{8}(18 + 0 - 0) = \frac{18}{8} = 2.25$$

substituting $x_1 = 2.25$, z = 0 in (2)

$$y_1 = \frac{1}{5}[3 - 2(2.25) + 2(0)] = -\ 0.3$$

Substituting $x_1 = 2.25$, $y_1 = -0.3$ in (3)

$$z_1 = \frac{1}{3}[6 + 2.25 + (-0.3)] = 2.65$$

$x_1 = 2.25$, $y_1 = -0.3$, $z_1 = -2.65$ are the first approximate values.

SECOND ITERATION

Substituting $y_1 = -\ 0.3$, $z_1 = 2.65$ in (1)

$$x_2 = \frac{1}{8}[18 + (-\ 0.3) - 2.65]$$

$$x_2 = 1.8813.$$

Substituting $x_2 = 1.8813$, $z_1 = 2.65$ in (2)

$$y_2 = \frac{1}{5}[3 - 2(1.8813) + 2(2.65)] = 0.9075.$$

Substituting $x_2 = 1.8813,\ y_2 = 0.9075$ in (3)

$$z_2 = \frac{1}{3}[6 + 1.8813 + 0.9075] = 2.9296$$

$$x_2 = 1.8813,\ y_2 = 0.9075,\ z_2 = 2.9296$$

are second approximate values of x, y, z.

THIRD ITERATION

Substituting $y_2 = 0.9075,\ z_2 = 2.9296$ in (1)

$$x_3 = \frac{1}{8}[18 + 0.9075, -2.9296] = 1.9972.$$

Substituting $x_3 = 1.9972,\ z_2 = 2.9296$ in (2)

$$y_3 = \frac{1}{5}\left[3 - 2(1.9972) + 2(2.9296)\right] = 0.9729.$$

Substituting $x_3 = 1.9972,\ y_3 = 0.9729$ in (3)

$$z_3 = \frac{1}{3}[6 + 1.9972 + 0.9729] = 2.9900$$

$\therefore \quad x_3 = 1.9972,\ y_3 = 0.9729,\ z_3 = 2.9900$

are third approximate values of x, y, z.

FOURTH ITERATION

Substituting $y_3 = 0.9729,\ z_3 = 2.9900$ in (1)

$$x_4 = \frac{1}{8}[18 + 0.9729 - 2.9900] = 1.9979.$$

Substituting $x_4 = 1.9979,\ z_3 = 2.9900$ in (2)

$$y_4 = \frac{1}{5}\left[3 - 2(1.9979) + 2(2.9900)\right] = 0.9968$$

Substituting $x_4 = 1.9979,\ y_4 = 0.9968$ in (3)

$$z_4 = \frac{1}{3}[6 + 1.9979 + 0.9968] = 2.9982$$

$$x_4 = 1.9979,\ y_4 = 0.9968,\ z_4 = 2.9982$$

are fourth approximate values of x, y, z.

FIFTH ITERATION:

Substituting $y_4 = 0.9968,\ z_4 = 2.9982$ in (1)

$$x_5 = \frac{1}{8}[18 + 0.9968] - 2.9982 = 1.9998.$$

Substituting $x_5 = 1.9998,\ z_4 = 2.9992$

$$y_5 = \frac{1}{5}[3 - 2(1.9998) + 2(2.9982)] = 0.9994.$$

Substituting $x_5 = 1.9998,\ y_5 = 0.9994$

$$z_5 = \frac{1}{3}[6 + (1.9998) + 0.9994] = 2.9997$$

$x_5 = 1.9998,\ y_5 = 0.9994,\ z_5 = 2.9997$

are fifth approximate values of x, y, z.

hence $x \simeq 2,\ \ y = 0.9994,\ \ z = 2.9999$ solution of given system.

EXAMPLE 46:

Solve by Gauss-Seidal method the equations

$$9x - 2y + z - t = 50$$

$$x - 7y + 3z + t = 20$$

$$-2x + 2y + 7z + 2t = 22$$

$$x + y - 2z + 6t = 18$$

SOLUTION

Rewriting the given equations

$$x = \frac{1}{9}(50 + 2y - z + t) \qquad \ldots(1)$$

$$y = \frac{1}{7}(20 + x - 3z - t) \qquad \ldots(2)$$

$$z = \frac{1}{7}(22 + 2x - 2y - 2t) \qquad \ldots(3)$$

$$t = \frac{1}{6}(18 - x - y + 2z) \qquad \ldots(4)$$

FIRST ITERATION

Substituting the initial values

$y = z = t = 0$ in (1) to find the first iteration

$$x_1 = \frac{1}{9}(50 + 2.0 - 0 + 0) = \frac{50}{9} = 5.5556.$$

Substituting $x_1 = 5.5556,\ z = 0,\ t = 0$ in (2)

$$y_1 = \frac{1}{7}[20 - 5.5556 + 3.0 - 0] = 2.0635.$$

Substituting $x_1 = 5.5556,\ y_1 = 2.0635,\ t = 0$ in (3)

$$z_1 = \frac{1}{7}\left[22 + 2(5.5556) - 2(2.0635) - 2(0)\right] = 4.1406.$$

Substituting $x_1 = 5.5556,\ y_1 = 2.0635,\ z_1 = 4.1406$ in (4)

$$t_1 = \frac{1}{6}\left[18 - 5.5556 - 2.0635 + 2(4.1406)\right] = 3.1104$$

$$x_1 = 5.5556,\ y_1 = 2.0635,\ z_1 = 4.1406,\ t_1 = 3.1104$$

are the first approximate values of x, y, z.

SECOND ITERATION

Substituting $y_1 = 2.0635,\ z_1 = 4.1406,\ t_1 = 3.1104$ in (1)

$$x_2 = \frac{1}{9}\left[50 + 2(2.0635) - 4.1406 + 3.1104\right] = 5.8996.$$

Substituting $x_2 = 5.8996,\ z_1 = 4.1406,\ t_1 = 3.1104$ in (2)

$$y_2 = -\frac{1}{7}\left[20 - 5.8996 + 3(4.1406) - 3.1104\right] = 3.3445.$$

Substituting $x_2 = 5.8996,\ y_2 = 3.3445,\ t = 3.1104$ in (3)

$$z_2 = \frac{1}{7}\left[22 + 2(5.8996) - 2(3.3445) - 2(3.1104)\right] = 2.9842.$$

Substituting $x_2 = 5.8996,\ y_2 = 3.3445,\ z_2 = 2.9842$ in (4)

$$t_2 = \frac{1}{6}\left[18 - 5.8996 - 3.3445 + 2(2.9842)\right] = 2.4541$$

$$x_2 = 5.8996,\ y_2 = 3.3445,\ z_2 = 2.9842,\ t_2 = 2.4541$$

are second approximate of values of x, y, z, t.

THIRD ITERATION

Substituting the $y_2 = 3.3445,\ z_2 = 2.9842,\ t_2 = 2.4541$ in (1)

$$x_3 = \frac{1}{9}\left[50 + 2(3.3445) - 2.9842 + 2.4541\right] = 6.2399.$$

Substituting $x_3 = 6.2399,\ z_2 = 2.9842,\ t_2 = 2.4541$ in (2)

$$y_3 = \frac{1}{7}\left[20 - 6.2399 + 3(2.9842) - 2.4541\right] = 2.8941.$$

Substituting $x_3 = 6.2399,\ y_3 = 2.8941,\ t_2 = 2.4541$ in (3)

$$z_3 = \frac{1}{7}\left[22 + 2(6.2399) - 2(2.8941) - 2(2.4541)\right] = 3.3976$$

Substituting $x_3 = 6.2399,\ y_3 = 2.8941,\ z_3 = 3.3976$ in (4)

$$t_3 = \frac{1}{6}\left[18 - 6.2399 - 2.8941 + 2(3.3976)\right] = 2.6102$$

$\therefore$ $x_3 = 6.2399,\ y_3 = 2.8941,\ z_3 = 3.3976,\ t_3 = 2.6102$

are the third approximate values of x, y, z similarly we can obtain.

FOURTH ITERATION

$$x_4 = 6.1112,\ y_4 = 3.0673,\ z_4 = 3.62668,\ t_3 = 2.5592$$

FIFTH ITERATION

$$x_5 = 6.1586,\ y_5 = 3.0118,\ z_5 = 3.3107,\ t_5 = 2.5752$$

SIXTH ITERATION

$$x_6 = 6.1431,\ y_6 = 3.0305,\ z_6 = 3.2964,\ t_6 = 2.5699$$

SEVENTH ITERATION

$$x_7 = 6.1483,\ y_7 = 3.0244,\ z_7 = 3.3011,\ t_7 = 2.5716$$

Now the values of sixth and seventh iteration values of x, y, x, t are nearly the same.

$\therefore$ The solution is $x = 6.14,\ y = 3.02,\ z = 3.30,\ t = 2.57$

Exercise 1(f)

Solve the following system of equations using Jacobi's as well as Gauss-Seidal iteration method.

1. $10x - y - z = 13$

   $x + 10y + z = 36$

   $-x - y + 10z = 35$

   (**Ans:** $x = 2,\ y = 3,\ z = 4$)

2. $x + 10y + z = 6$

   $10x + y + z = 6$

   $x + y + 10z = 6$

   (**Ans:** $x = y = z = 0.5$)

3. $2x + y + z = 4$

   $x + 2y + z = 4$

   $x + y + 2z = 4$

   (**Ans:** $x = 1,\ y = 1,\ z = 1$)

4. $4x + 2y + z = 14$

   $x + 5y - z = 10$

   $x + y + 8z = 20$

   (**Ans:** $x = 2,\ y = 1.99,\ z = 1.999$)

5. $10x_1 - 5x_2 - 2x_3 = 3$

   $4x_1 - 10x_2 + 3x_3 = -3$

   $x_1 + 6x_2 + 10x_3 = -3$

   (**Ans:** $x_1 = 0.342,\ x_2 = 0.285,\ x_3 = -0.505$)

6. $8x + y + z = 8$

$2x + 4y + z = 4$

$x + 3y + 5z = 5$

(**Ans:** $x = 0.876,\ y = 0.419,\ z = 0.574$)

7. $3x + 4y + 15z = 54.8$

$x + 12y + 3z = 39.66$

$10x + y - 2z = 7.74$

(**Ans:** $x = 1.075,\ y = 2.524,\ z = 2.765$)

8. $10x_1 - 2y_2 - x_3 - x_4 = 3$

$-2x_1 + 10x_2 - x_3 - x_4 = 15$

$-x_1 - x_2 + 10x_3 - 2x_4 = 27$

$-x_1 - x_2 - 2x_3 + 10x_4 = -9$

(**Ans:** $x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 0$)

9. $15x + 3y - 2z = 85$

$2x + 10y + z = 51$

$x - 2y + 8z = 5$

(**Ans:** $x = 5,\ y = 4,\ z = 1$)

10. $44x + y + z = 100$

$-x + 6y + 25z = 80$

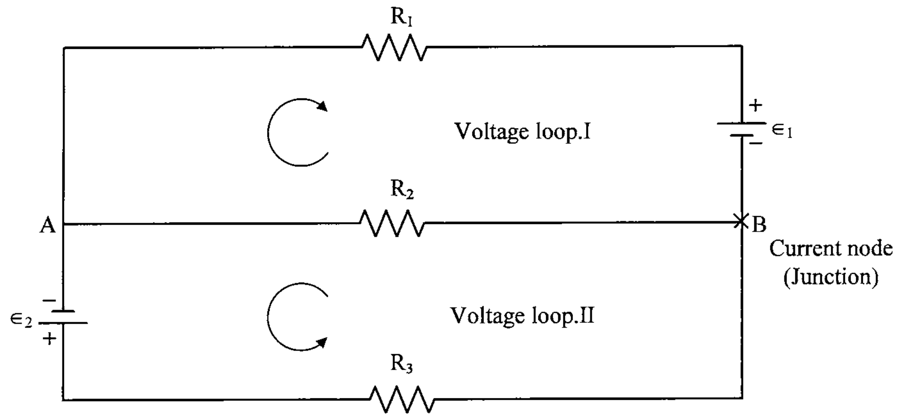
$x + 20y + z = 70$

(**Ans:** $x = 2.14,\ y = 3.26,\ z = 2.50$)

1.6 Finding Current in a Electrical Circuit

Introduction: A simple electric circuit is a closed connection of batteries, resistances, wires. An electric circuit consists of voltage loops current nodes.

A loop is any closed path in the circuit (network). A junction (node) in an electrical circuit is a point where three or more components like resistances and batteries (cells) meet.



The following physical quantities are measured in the electrical circuits.

(i) **Current:** Current is denoted by i, measured in amps (A)

(ii) **Resistance:** Resistance is denoted by R, measured in ohms (Ω)

(iii) **Electrical potential difference:** Denoted by V measured in volts (V)

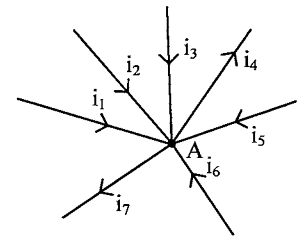
Three basic laws govern the flow of current in the electrical circuit. Ohm's law is useful in the analysis of simple circuits and two laws of kirchoff are useful to find the currents that flow in complex circuits.

(i) *Kirchoff's first law* (is also called current law) states that the algebraic sum of electric currents at a junction in a circuit is zero.

From the figure the current flow at the junction A can be written as

$$i_1 + i_2 + i_3 - i_4 + i_5 + i_6 - i_7 = 0$$

Here the sign convention is that the current flowing into junction is positive and current flowing out is negative



(ii) *Kirchoff's second law* (is also called voltage law) states that the algebraic sum of changes in potential around any closed loop is zero (this is often is called loop theorem)

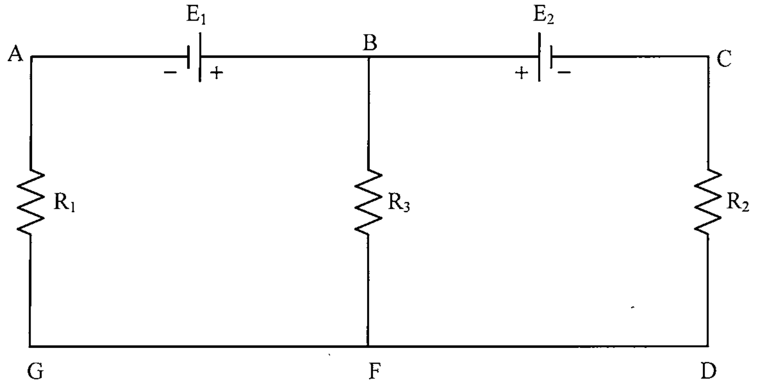
i.e., $\sum V = 0,$

It can be expressed in terms of potential drops and emfs as $\sum iR + \sum E = 0$

The sign convention is that while following are specific direction to traverse the loop, potential drops along this direction are to be taken with negative sign, and potential rises along this direction are to be taken with positive sign.

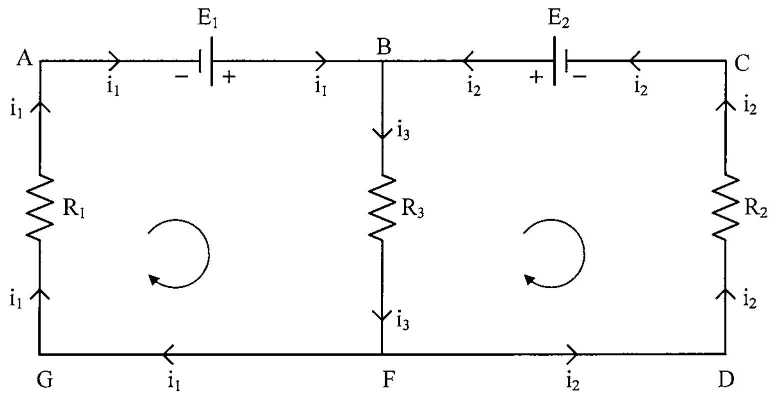
1.6.1 Current in multi-loop circuit

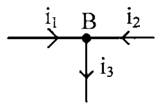
The resistances R_1, R_2, R_3 and cells of emfs E_1 and E_2 (having no internal resistance) are connected as shown in figure, find the currents flowing in the circuit.



Multi-loop Circuit

Let i_1, i_2, and i_3 be the current flowing along FGAB, FDCB and BF respectively as shown in figure





Appling kirchoff's current law at the junction B.

$$i_1 + i_2 = i_3$$

$$\therefore \quad i_1 + i_2 - i_3 = 0 \qquad .....(1)$$

Now consider the direction of flow in loop ABFGA, the currents i_1 and i_3 are in the clockwise direction, i.e., A to B to F to G to A and traversing $\in_1$ from negative to positive terminal. Therefore E_1 is positive.

Applying kirchoff's voltage law for the loop ABFGA

$$E_1 - i_3R_3 - i_1R_1 = 0$$

$\therefore$ $$i_1R_1 + i_3R_3 = E_1 \qquad .....(2)$$

Similarly for the loop BCDFB, directions of i_2, i_3 are anticlockwise directions and $\in_2$ traversing from positive to negative

$\therefore$ E_2 is negative

using kirchoff's law for the loop BCDFB

$$E_2 + i_2R_2 + i_3R_3 = 0$$

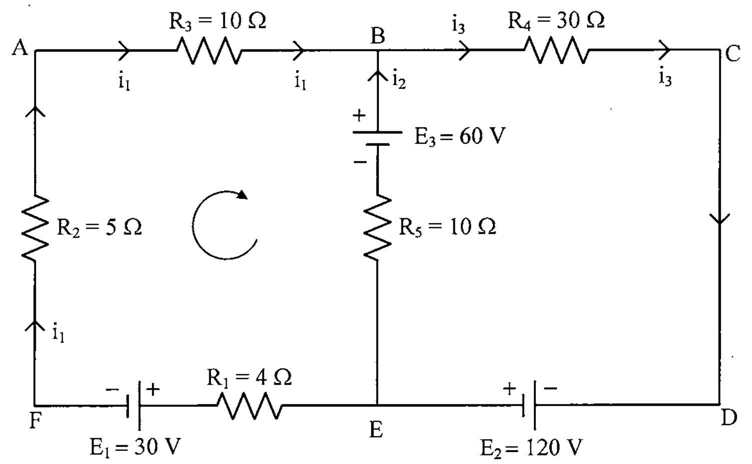
$\therefore$ $$i_2R_2 + i_3R_3 = E_2 \qquad .....(3)$$

From equations (1), (2) and (3) using matrix method, solve linear system of equations, to get the current values if i_1, i_2, i_3 from the matrix equation

$$\begin{pmatrix} 1 & 1 & -1 \\ R_1 & 0 & R_3 \\ 0 & R_2 & R_3 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ E_1 \\ E_2 \end{pmatrix}$$

EXAMPLE 47

Find the currents in the circuit for the following net work.



SOLUTION

Let i_1, i_2 and i_3 be the currents flowing in the loop EFABE, path EB and the loop BCDEB respectively electric current at the junction B (node) applying kirchoff's first law

$$i_1 + i_2 = i_3$$

$$i_1 + i_2 - i_3 = 0 \quad .....(1)$$

From the figure $E_1 = 30V$, $E_2 = 120V$, $E_3 = 60V$ and $R_1 = 4\Omega$ $R_2 = 5\Omega$ $R_3 = 10\Omega$ $R_4 = 30\Omega$

Applying kirchoff's second loop to the loop EFABE

$$-i_1R_1 - E_1 - i_1R_2 - i_1R_3 - E_3 + i_2R_5 = 0$$

$$-4i_1 - 30 - i_1 5 - 10i_1 - 60 + 10\, i_2 = 0$$

$$-19i_1 + 10i_2 = 90 \quad .....(2)$$

using kirchoff's second law to the loop BCDEB.

$$-i_3R_4 + E_2 - i_2R_5 + E_3 = 0$$

$$-30i_3 + 120 - 10i_2 + 60 = 0$$

$$i_2 + 3i_3 = 18 \quad .....(3)$$

To find the currents i_1, i_2 and i_3 solve the equations (1), (2) and (3) using matrix method. The matrix equation is

$$\begin{pmatrix} 1 & 1 & -1 \\ -19 & 10 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 90 \\ 18 \end{pmatrix}$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -19 & 10 & 0 & 90 \\ 0 & 1 & 3 & 18 \end{array}\right) \quad R_2 \to R_2 + 19\, R_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 29 & -19 & 90 \\ 0 & 1 & 3 & 18 \end{array}\right) \quad R_2 \leftrightarrow R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & 18 \\ 0 & 29 & -19 & 90 \end{array}\right) \quad R_3 \to R_3 - 29\, R_2$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & 18 \\ 0 & 0 & -106 & -432 \end{array}\right)$$

$\Rightarrow$

$$i_1 + i_2 - i_3 = 0$$

$$i_2 + 3i_3 = 18$$

$$-106i_3 = -432$$

$$i_3 = 4.0754 \text{ A}$$

$$i_2 = 18 - 3i_3 = 18 - 3(4.0754)$$

$$i_2 = 5.7738 \text{ A}$$

$$i_1 = i_3 - i_2 = 4.0754 - 5.7738$$

$$i_1 = -1.6984 \text{ A}$$

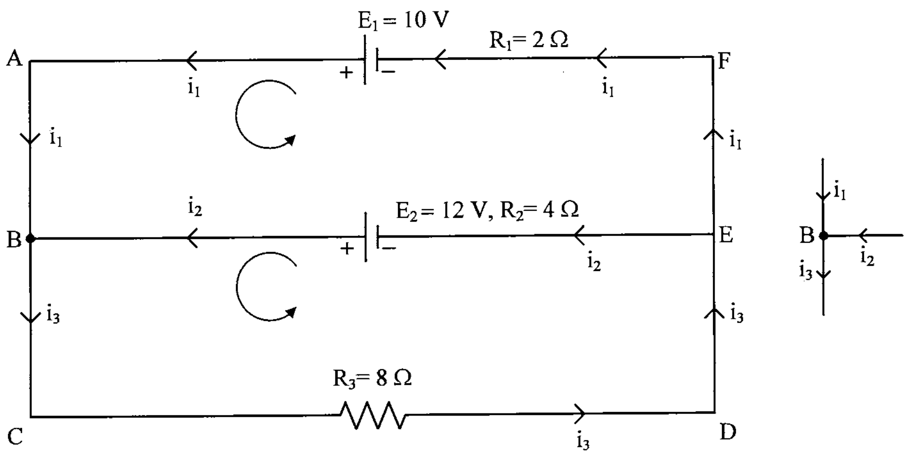
$$i_2 = 5.7738 \text{ A}$$

$$i_3 = 4.0754 \text{ A}$$

Solved Examples

EXAMPLE 48

Find the current flowing in given network.



SOLUTION

Given cell E_1 with emf 10 V and internal resistance 2 Ω

i.e., $E_1 = 10$ V, $R_1 = 2$ Ω and $E_2 = 12$ V and internal resistance $R_2 = 4$ Ω, $R_3 = 8$ Ω

Using kirchoff's first law, electric current at junction B is

$$i_1 + i_2 = i_3$$

$$i_1 + i_2 - i_3 = 0 \qquad .....(1)$$

Applying kirchoff's second law to loop A B E F A

$$-E_2 + i_2 R_2 - i_1 R_1 + E_1 = 0$$

(E_2 traversing from positive to negative, hence E_2 is negative, E_1 traversing from negative to positive, E_1 direction of i_2 is clockwise and direction of i_1 is anti clockwise direction)

From the figure substituting the values in above equation

$$-12 + 4\, i_2 - 2\, i_1 + 10 = 0$$

$$i_1 - 2\, i_2 = -1 \qquad .....(2)$$

Similarly, applying kirchoff's law to the loop B C D E B

$$-i_3 R_3 - i_2 R_2 + E_2 = 0$$

$$-8\, i_3 - 4\, i_2 + 12 = 0$$

$$i_2 + 2\, i_3 = 3$$

To minimize the number of equations, substituting $i_3 = i_1 + i_2$ from equation (1)

$$i_2 + 2(i_1 + i_2) = 3$$

$$2\, i_1 + 3\, i_2 = 3 \qquad .....(3)$$

Solving equations (2) and (3) using matrix method

$$\begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Augmented matrix is

$$\left(\begin{array}{cc|c} 1 & -2 & -1 \\ 2 & 3 & 3 \end{array}\right) \text{ using } R_2 \to R_2 - 2R_1$$

$$\left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 7 & 5 \end{array}\right)$$

$$i_1 - 2\, i_2 = -1$$

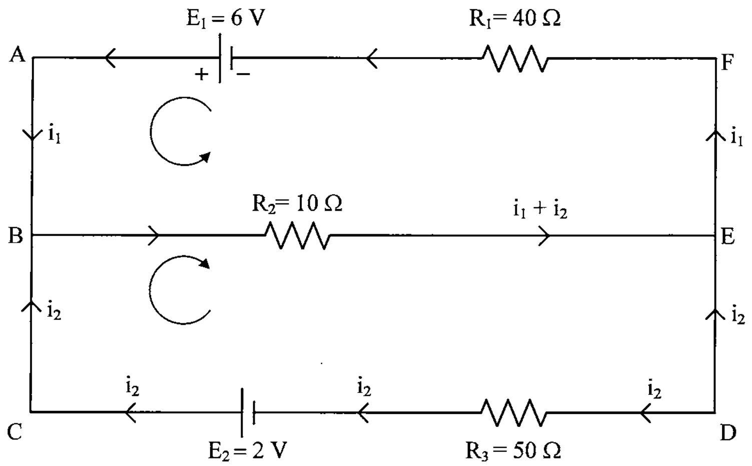
$$7\,i_2 = 5$$

$$\therefore \quad i_2 = \frac{5}{7} = 0.71$$

and $\quad i_1 = 0.42$

EXAMPLE 49

Find the current in each cell considering the circuit given in the figure.



SOLUTION

Applying kirchoff's voltage law to the loop A B E F A.

$$E_1 - i_1 R_1 - (i_1 + i_2)R_2 = 0$$

$$6 - 40\,i_1 - (i_1 + i_2)10 = 0$$

$$25\,i_1 + 5\,i_2 = 3 \qquad \text{.....(1)}$$

Using kirchoff's second law to the loop B E D C B

$$E_2 - (i_1 + i_2)R_2 - i_2 R_3 = 0$$

$$2 - (i_1 + i_2)10 - 50 i_2 = 0$$

$$5\,i_1 + 30\,i_2 = 1 \qquad \text{.....(2)}$$

Solving the equations (1) and (2) using matrix method

$$\begin{pmatrix} 25 & 5 \\ 5 & 30 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Augmented matrix is

$$\left(\begin{array}{cc|c} 25 & 5 & 3 \\ 5 & 30 & 1 \end{array}\right) \quad R_2 \to R_2 - \frac{1}{5}R_1$$

$$\left(\begin{array}{cc|c} 25 & 5 & 3 \\ 0 & 29 & \frac{2}{5} \end{array}\right)$$

$$25\, i_1 + 5\, i_2 = 3$$

$$29\, i_2 = \frac{2}{5}$$

$$i_2 = \frac{2}{145} = 0.0138 \text{ Amp}$$

$$25\, i_1 + 5\left(\frac{2}{145}\right) = 3$$

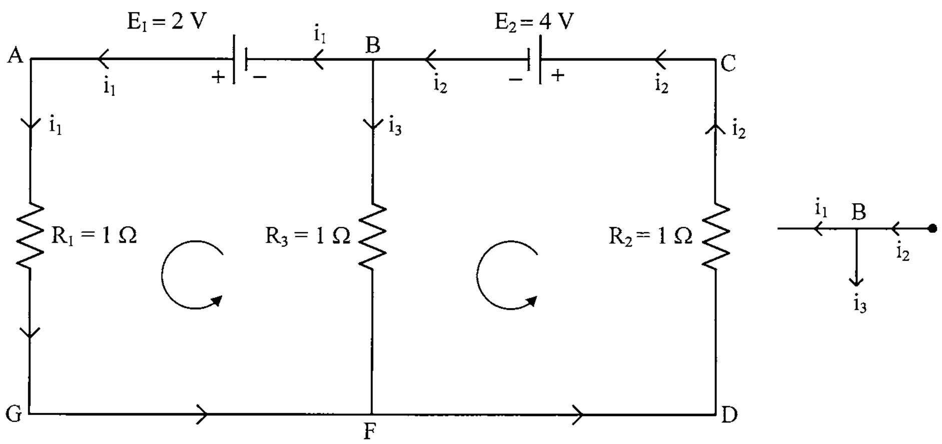
$$i_1 = \frac{17}{145}$$

$$\therefore \quad i_1 = 0.1172\ \text{A}$$

$$i_2 = 0.0138\ \text{A}$$

EXAMPLE 50

In a given circuit, the values are as following $E_1 = 2$ V, $E_2 = 4$ V, $R_1 = R_2 = R_3 = 1\ \Omega$. Calculate the values of the currents i_1, i_2, i_3



SOLUTION

Given the values $E_1 = 2$ V, $E_2 = 4$ V, $R_1 = R_2 = R_3 = 1\ \Omega$

From the loop B A G F B using kirchoff's second law

$$E_1 - i_1 R_1 + i_3 R_3 = 0$$

given $(R_1 = R_2 = R_3 = 1\ \Omega)$

$$2 - i_1 + i_3 = 0$$

$$-i_1 + i_3 = -2 \qquad \text{.....(1)}$$

Applying kirchoff's second law to the loop B F D C B

$$-i_3 R_3 - i_2 R_2 - E_2 = 0$$

$$-i_2 - i_3 - 4 = 0 \qquad (R_1 = R_2 = R_3 = 1\ \Omega)$$

$$i_2 + i_3 = -4 \qquad \text{.....(2)}$$

At the junction B applying kirchoff's first law

$$-i_1 + i_2 - i_3 = 0$$

$$i_1 - i_2 + i_3 = 0 \qquad \text{.....(3)}$$

Solving the system of equations (1), (2) and (3) from the matrix equation

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -4 \end{array}\right) \qquad R_2 \to R_2 + R_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 1 & 1 & -4 \end{array}\right) \qquad R_3 \to R_3 + R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 3 & -6 \end{array}\right)$$

$\Rightarrow \quad i_1 - i_2 + i_3 = 0$

$-i_2 + 2\, i_3 = -2$

$3\, i_3 = -6$

$i_3 = -2 \text{ A}$

$-i_2 + 2(-2\text{A}) = -2$

$i_2 = -2 \text{ A}$

Substituting i_2, i_3 values in $i_1 - i_2 + i_3 = 0$

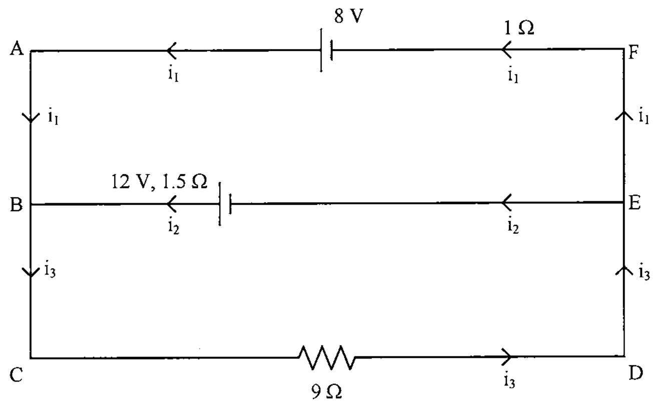
$i_1 = 0$

$\therefore$ The required current values are

$i_1 = 0$, $i_2 = -2$ A, $i_3 = -2$ A

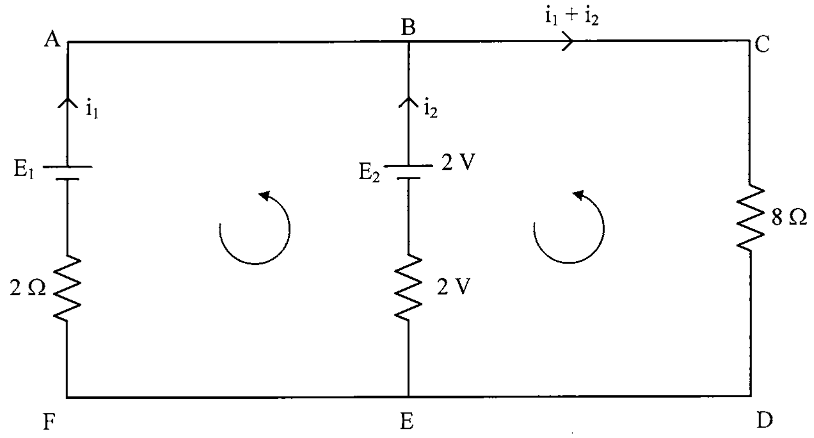
Exercise 1(g)

1. Solve for current values i_1, i_2, i_3 from the figure



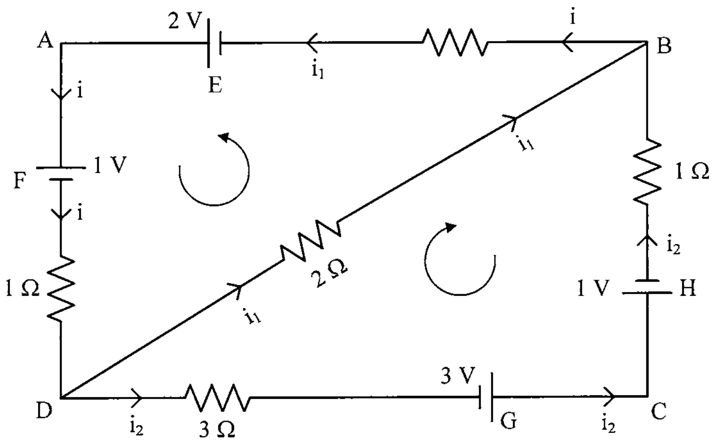
Ans: $i_1 = -1$ A, $i_2 = 2$ A, $i_3 = 1$A

2. Find the values of the current flowing in the circuit given below



Ans: $i_1 = \frac{2}{3}$ A, $i_2 = \frac{1}{3}$ A

3. Find the current through the 2 Ω resister across DB in the (figure) given circuit. Cell E is of emf 2 V and internal resistance 2 Ω. Cell F is of emf 1 V and internal resistance 1 Ω. Cell G is of emf 3 V and internal resistance 3 Ω and cell H is of emf 1 V and internal resistance.



Ans: $i_1 = \frac{-1}{13}$, $i_2 = \frac{6}{13}$, $i = \frac{5}{13}$

Summary

1. *Rank of a matrix:* The rank of a matrix is the order 'r' of the largest non-vanishing minor of the matrix.
2. *Elementary transformation of a matrix:*

   (a) *Row transformation:*

   (i) Interchange of i^{th} and j^{th} row $\rightarrow R_{ij}$

   (ii) Multiplication of i^{th} row by non-zero scalar 'l' $\rightarrow R_{i(l)}$

   (iii) Addition of 'l' times the elements of j_{th} row to corresponding elements of i_{th} row–$R_{ij(l)}$.

   (b) Column transformations are similar to (a) –replace 'R' by 'C'
3. *Computation of Rank of matrix:*

   Method 1:

   Echelon form: Transform the given matrix to an 'echelon form' using elementary transformations. The rank of the matrix is equal to the number of non-zero rows of echelon form.

   Method 2:

   Canonical form or Normal form: Reduce the given matrix 'A' to one of the normal forms $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $[I_r]$, $[I_r 0]$, or $\begin{bmatrix} Ir \\ 0 \end{bmatrix}$, using elementary transformation, Then Rank of

   $A = r$,
4. Simultaneous Linear Equations - Methods of solution.

   1. System for 'm' equation in n unknowns can be written in a matrix form as

   $$AX = B, \text{ where, } A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}; \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}$$

   If $b_i = 0$, the system is homogeneous i.e. B = 0, Otherwise, it is non-homogeneous.

   2. *Condition for the consistency:* A system of linear equations AX = B is consistent iff the rank of A is equal to the rank of the augmented matrix [A/B].

3. Solution of AX = B – Working rule.

(i) Find r(A) and r(A/B) by applying elementary transformations.

(ii) If $r(A) = r(A/B) = n$, (n, being number of unknowns). The system is consistent and has unique solution. [In this case $|A| \neq 0$]

(iii) If $r(A) = r(A/B) < n$, the system is consistent and has infinite number of solutions.

(iv) If $r(A) \neq r(A/B)$, the system is inconsistent and has no solution.

4. *Other method :* Let A be a 3×3 matrix then,

(i) *Matrix inversion method:* AX = B has the solution, $X = A^{-1} B$. (if $|A| \neq 0$)

(ii) *CRAMER'S rule [Method of determinants]:* Let $|A| \neq 0$. Let $\Delta = |A|$; we obtain 3 more determinants Δ_1, Δ_2, Δ_3, of 3 matrices obtained by replacing the 1st, 2nd, and 3rd columns of A by the column matrix 'B' of the system respectively. Then $x_1 = \frac{\Delta_1}{\Delta}$; $x_2 = \frac{\Delta_2}{\Delta}$; $x_3 = \frac{\Delta_3}{\Delta}$

(iii) *Gauss-Jordan Method:* Reduce the augmented matrix (A/B) to the form $[I_3 X]$ where I_3 is unit matrix. Then $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is solution.

5. *Gauss-Elimination method:*

Step 1: Eliminate the unknowns $x_1, x_2, \ldots x_{n-1}$ successively and obtain an upper triangular matrix.

Step 2: The last equation of this matrix gives the value of x_n.

Step 3: The back substitution of unknowns in the other equations give the other unknowns.

6. *LU Decomposition method:* Solution of AX = B

(i) Let all principal minors of A be non-singular.

(ii) $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & l & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ Then A = LU LUX = B:

Let LY = B where $y = [y_1 \quad y_2 \quad y_3]^1$ and Y = UX.

(iii) From LU = A, we get both L and U.

(iv) From LY = B, we get y_1, y_2, y_3 ; and then

(v) From UX = Y, we get x_1, x_2, x_3.

7. *Tri-diagonal matrix:* Matrices of the type $\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$ are tri diagonal matrices.

8. Solution of Tri-diagonal systems – see 1.4.11.
9. Gauss-Siedel Iterative Method
10. Finding Current in a Electrical Circuit
11. Homogeneous linear equations : AX = 0, where $A = (a_{ij})_{m \times n}$; $X = [x_1, x_2 \ldots x_n]^1$;

Solution of AX = 0 can be done by elementary transformations.

Conclusions :

(i) The system AX = 0 is always consistent since the trivial solution

$x_1 = x_2 = \ldots = x_n = 0$ always exists.

(ii) If rank of (A/B) = rank of A = n [|A| $\neq$ 0], then the trivial solution is the only solution.

(iii) If rank of (A/B) = rank of A = $r < n$, [|A| $\neq$ 0] the solution has infinite number of non-trivial solutions involving (n – r) arbitrary constants.

Solved University Questions

1. Reduce the matrix A to its normal form where $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ and hence find its rank.

[JNTU 2006, 2006]

Solution

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix} (c_1 \leftrightarrow c_2) \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 2 & 6 \\ 1 & 1 & 3 & 1 \end{bmatrix} \left(c_2 \rightarrow \frac{1}{2} c_2 \right)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 3 & 1 \end{bmatrix} \left(R_2 \rightarrow \frac{1}{2} R_2 \right) \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{bmatrix} (R_3 \rightarrow R_3 - R_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} (R_3 \rightarrow R_3 - R_2) \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} c_3 = c_3 - c_2 \\ c_4 = c_4 - 3c_2 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} c_3 = c_3 - 2c_1 \\ c_4 = c_4 + 2c_1 \end{Bmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$, which is the normal form.

Hence rank of A = 2 .

2. Show that the only real value of λ for which the following equations have non-trivial solution is 6 and solve them when

$\lambda = 6 \; x + 2y + 3z = \lambda x \; ; \; 3x + y + 2z = \lambda y \; ; \; 2x + 3y + z = \lambda z$

[JNTU 2002, 2005, 2006]

Solution

The given system can be written as $\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ....(1)

Which is a homogeneous system. It has a non – trivial solution if the coefficient matrix is singular i.e., if its determinant is zero.

i.e., $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 3\lambda^2 - 15\lambda - 18 = 0$

$\Rightarrow (\lambda-6)(\lambda^2+3\lambda+3)=0$

$\Rightarrow \lambda = 6$ is the only real value $\left[\text{Since } \lambda^2+3\lambda+3=0 \text{ has complex roots.}\right]$

$\therefore$ From (1), the coefficient matrix,

$$A = \begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix} R_1 \leftrightarrow R_3 \sim \begin{bmatrix} 2 & 3 & -5 \\ 3 & -5 & 2 \\ -5 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -5 \\ 3 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} (R_3 = R_3 + R_2 + R_1) \sim \begin{bmatrix} 2 & 3 & -5 \\ 1 & -8 & 7 \\ 0 & 0 & 0 \end{bmatrix} (R_3 = R_2 \text{-} R_1)$$

$$\sim \begin{bmatrix} 0 & 19 & -19 \\ 1 & -8 & 7 \\ 0 & 0 & 0 \end{bmatrix} (R_1 = R_1 \text{-} 2R_2) \sim \begin{bmatrix} 0 & 1 & -1 \\ 1 & -8 & 7 \\ 0 & 0 & 0 \end{bmatrix} \left(R_1 = \frac{1}{19}R_1\right)$$

$$\sim \begin{bmatrix} 1 & -8 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} (R_1 \leftrightarrow R_2); \quad \begin{vmatrix} -8 & 7 \\ 1 & -1 \end{vmatrix} \neq 0$$

$\therefore$ rank of A= 2 < 3 (no. of unknowns)

The system has infinite no. of solutions and involves 3 – 2 = 1 arbitrary constant $'\alpha'$ say.

Hence the system reduces to,

$x - 8y + 7z = 0$; $y - z = 0$; Taking $z = \alpha$ we have $x = y = z = \alpha$ as the solution of the system.

3. Prove that the following set of equations are consistent and solve them.

$3x + 3y + 2z = 1$; $x + 2y = 4$; $10y + 3z = -2$; $2x - 3y - z = 5$ [JNTU 2006]

Solution

The given set of the equations can be put in Matrix from AX= B as,

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix} \qquad \ldots..(1)$$

Where $A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$

Augmented matrix

$$(A,B) = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 = R_2 - 3R_1 \\ R_4 = R_4 - 4R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix} R_2 = -\frac{1}{3} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & +11/3 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & -7 & -1 & -3 \end{bmatrix} R_3 = R_3 - 10R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -4 \\ 0 & 7 & 1 & 3 \end{bmatrix} \quad \begin{array}{l} R_3 = R_3 \times \frac{3}{29} \\ R_4 = -R_4 \end{array}$$

$$\sim\begin{bmatrix}1 & 2 & 0 & 4\\ 0 & 1 & -2/3 & 11/3\\ 0 & 0 & 1 & -4\\ 0 & 0 & 17/3 & -68/3\end{bmatrix} R_4 = R_4 - 7R_2 \ \sim\begin{bmatrix}1 & 2 & 0 & 4\\ 0 & 1 & -2/3 & 11/3\\ 0 & 0 & 1 & -4\\ 0 & 0 & 1 & -4\end{bmatrix} \sim R_4 = R_4 \times \frac{3}{17}$$

$$\sim\begin{bmatrix}1 & 2 & 0 & 4\\ 0 & 1 & -2/3 & 11/3\\ 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 0\end{bmatrix} R_4 = R_4 - R_3 \qquad ....(2)$$

Which shows that $r(\text{A,B}) = \text{r}(\text{A}) = 3$, the no. of unknowns.

$\therefore$ The given set of equations are consistent.

Further , from the reduced form (2) of (A,B) , we get,

$$x + 2y = 4 \ ; \ y - \frac{2}{3}z = \frac{11}{3} \ ; \ z = -4$$

$$\therefore \qquad y = \frac{11}{3} + \frac{2}{3}(-4) = 1 \ ; \ x + 2 = 4 \Rightarrow x = 2$$

$$\therefore \qquad x = 2, \quad y = 1, \quad z = -4$$

4. Find the LU – decomposition of the matrix 'A' and solve the system AX= B where

$$\begin{bmatrix}-3 & 12 & -6\\ 1 & -2 & 2\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-33\\ 7\\ -1\end{bmatrix}$$ [JNTU 2006]

Solution

Let $$\text{A} = \begin{bmatrix}-3 & 12 & -6\\ 1 & -2 & 2\\ 0 & 1 & 1\end{bmatrix} = \text{LU}, \text{ where;}$$

$$\text{L} = \begin{bmatrix}1 & 0 & 0\\ \ell_{21} & 1 & 0\\ \ell_{31} & \ell_{32} & 1\end{bmatrix} \text{ and U} = \begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

i.e., $u_{11} = -3,\ u_{12} = 12,\ u_{13} = -6$;

$\ell_{21}u_{11} = 1 \Rightarrow \ell_{21} = -1/3$; $\quad \ell_{31}u_{11} = 0 \Rightarrow \ell_{31} = 0$;

$\ell_{21}u_{12} + u_{22} = -2 \Rightarrow -1/3.12 + u_{22} = -2 \Rightarrow u_{22} = 2$

$\ell_{21}u_{13} + u_{23} = 2 \Rightarrow -1/3(-6) + u_{23} = 2 \Rightarrow u_{23} = 0$

$\ell_{31}u_{12} + \ell_{32}u_{22} = 1 \Rightarrow (0)(12) + 2\ell_{32} = 1 \Rightarrow \ell_{32} = \frac{1}{2}$

$\ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} = 1 \Rightarrow 0 + 0 + u_{33} = 1 \Rightarrow u_{33} = 1$

$\therefore$ The given system becomes L U X = B

i.e.,
$$\begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 12 & -6 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix} \quad \text{.....(1)}$$

Where
$$\begin{bmatrix} -3 & 12 & -6 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \text{.....(2)}$$

$(1) \Rightarrow y_1 = -33; \quad -\frac{1}{3}y_1 + y_2 = 7 \Rightarrow y_2 = -4; \quad \frac{1}{2}y_2 + y_3 = -1 \Rightarrow y_3 = 1$

$\therefore (2) \Rightarrow -3x + 12y - 6z = y_1 = -33; \quad 2y = y_2 = -4; \quad z = y_3 = 1$

Back substituting gives ,

$z = 1, y = -2, \quad x = 1 \Rightarrow x = 1, \quad y = -2$ and $z = 1$

5. Define and find the rank of A where $A=\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

[JNTU 2005, 2005S, 2006]

Solution

For definition of rank, see 1.2.3

$$A=\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 4R_1 \\ R_4 = R_4 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 = R_3 - R_2 \\ R_4 = R_4 - 3R_2 \end{matrix}$$

Since the reduced matrix has 2 non – zero rows, rank of A = 2

6. Determine whether the following system will have a non – trivial solution. If so, solve them.

$4x + 2y + z + 3w = 0;\ \ 6x + 3y + 4z + 7w = 0;\ \ 2x + y + w = 0$

[JNTU 2004, 2006]

Solution

Writing the given system in the matrix from, AX = 0, we get,

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0\ ;\ \ \therefore A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 6 & 3 & 4 & 7 \\ 4 & 2 & 1 & 3 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_2 = \frac{1}{4} R_2$$

$$\sim\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 = R_3 - R_2 \text{ ; Since the reduced matrix has} \qquad .....(1)$$

2 non – zero rows , rank of A = 2< 4 (no. of unknowns).

$\therefore$ The system has infinite numbers of solutions involving 4 – 2 = 2 arbitrary constants.

From (1) we get , $2x + y + w = 0, \quad z + w = 0$

Taking $y = \alpha, \quad w = \beta$ $(\alpha,\beta$ being arbitrary constants$)$, we get ,

$$z = -\beta, \; 2x + \alpha + \beta = 0 \Rightarrow \; x = -\frac{1}{2}(\alpha + \beta)$$

Hence the solution of the system is , $x = -\frac{1}{2}(\alpha + \beta)$; $y = \alpha$; $z = -\beta$ and $w = \beta$.

7. Find the rank of the matrix $A=\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ by reducing it to the normal form.

[JNTU 2006S]

Solution

$$A=\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} \quad R_2 = R_2 - 4R_1; R_3 = R_3 - 2R_1; R_4 = R_4 - R_1$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} \quad \begin{matrix} c_2 = c_2 + c_1 \\ c_4 = c_4 - 3c_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \end{bmatrix} R_2 \leftrightarrow R_4$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & -1 & 1\\0 & 0 & 0 & 0\\0 & 6 & 0 & -10\end{bmatrix} R_2=-R_2 \sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & -1 & 1\\0 & 0 & 0 & 0\\0 & 0 & 6 & -16\end{bmatrix} R_4=R_4-6R_2$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 6 & -16\end{bmatrix} \begin{matrix}c_3=c_3+c_2\\c_4=c_4-c_2\end{matrix} \sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 6 & -16\\0 & 0 & 0 & 0\end{bmatrix} R_3 \leftrightarrow R_4$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & -8/3\\0 & 0 & 0 & 0\end{bmatrix} R_3=\frac{1}{6}R_3 \sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix} c_4=c_4+\frac{8}{3}c_3$$

$\sim\begin{bmatrix}I_3 & 0\\0 & 0\end{bmatrix}$ which is the normal form.

$\therefore$ Rank of A = 3.

8. Test for consistency the set of equations and solve them if they are consistent.

$$x+2y+2z=2;\ \ 3x-2y-z=5;\ \ 2x-5y+3z=-4;\ \ x+4y+6z=0$$

[JNTU 2006S, 2005S]

Solution

Writing the set of equations in the matrix form AX = B, we get,

$$A=\begin{bmatrix}1 & 2 & 2\\3 & -2 & -1\\2 & -5 & 3\\1 & 4 & 6\end{bmatrix};\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}\quad;\quad B=\begin{bmatrix}2\\5\\-4\\0\end{bmatrix}$$

Augmented matrix $(A,B)=\begin{bmatrix}1 & 2 & 2 & 2\\3 & -2 & -1 & 5\\2 & -5 & 3 & 4\\1 & 4 & 6 & 0\end{bmatrix}$

$$\sim\begin{bmatrix}1 & 2 & 2 & 2\\0 & -8 & -7 & -1\\0 & -9 & -1 & -8\\0 & 2 & 4 & -2\end{bmatrix} R_2 = R_2 - 3R_1;\ R_3 = R_3 - 2R_1;\ R_4 = R_4 - R_1$$

$$\sim\begin{bmatrix}1 & 2 & 2 & 2\\0 & 8 & 7 & 1\\0 & 9 & 1 & 8\\0 & 1 & 2 & -1\end{bmatrix} R_2 = -R_2;\ R_3 = -R_3;\ R_4 = \frac{1}{2}R_4$$

$$\sim\begin{bmatrix}1 & 2 & 2 & 2\\0 & 1 & 2 & -1\\0 & 9 & 1 & 8\\0 & 8 & 7 & 1\end{bmatrix} R_2 \leftrightarrow R_4 \sim\begin{bmatrix}1 & 2 & 2 & 2\\0 & 1 & 2 & -1\\0 & 0 & -17 & 17\\0 & 0 & -9 & 9\end{bmatrix} \begin{matrix}R_3 = R_3 - 9R_2\\R_4 = R_4 - 8R_2\end{matrix}$$

$$\sim\begin{bmatrix}1 & 2 & 2 & 2\\0 & 1 & 2 & -1\\0 & 0 & 1 & -1\\0 & 0 & 1 & -1\end{bmatrix} \begin{matrix}R_3 = -\frac{1}{17}R_3\\R_4 = -\frac{1}{9}R_4\end{matrix} \sim\begin{bmatrix}1 & 2 & 2 & 2\\0 & 1 & 2 & -1\\0 & 0 & 1 & -1\\0 & 0 & 0 & 0\end{bmatrix} R_4 = R_4 - R_3$$

Which is in Echelon form with 3 non – zero rows.

$\therefore$ Rank of (A ,B) = Rank of A = 3.

Hence the system is consistent. It is equivalent to

$$\left.\begin{matrix}x+2y+2z=2\\y+2z=-1\\z=-1\end{matrix}\right\}\Rightarrow y=1\ ;\ x=2 \qquad \therefore\ x=2\ ,\ y=1\ ,\ z=-1$$

9. Determine whether the following equations will have a non – trivial solution. If so solve them.

$x+y-2z+3w=0;\ x-2y+z-w=0;\ 4x+y-5z+8w=0$

$5x-7y+2z-w=0$ [JNTU 2006]

Solution

Writing the given system in matrix form AX = 0, we have ,

$$A = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix} \quad \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - 4R_1 \\ R_4 = R_4 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \end{bmatrix} \quad R_4 = R_4 \times \frac{1}{4}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 = R_3 - R_2; \; R_4 = R_4 - R_2$$

$\therefore$ Rank of A = 2 < 4 (no. of unknowns)

$\therefore$ The system has infinite number of non – trivial solutions involving 4 – 2 = 2 arbitrary constants.

We have, x+y-2z+3w=0; -3y+3z-4w=0

Taking $z = c_1$, $w = c_2$ we get, $y = \frac{1}{3}(3c_1 - 4c_2)$

$$\therefore \; x = -\frac{1}{3}[3c_1 - 4c_2] + 2c_1 - 3c_2 = c_1 - \frac{5}{3}c_2$$

$\therefore$ The solution of the given system is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} c_1\text{-}(5/3) \;\; c_2 \\ c_1\text{-}(4/3) \;\; c_2 \\ c_1 \\ c_2 \end{bmatrix}$$

10. Find the value of 'k' such that the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2. [JNTU 2006]

Solution

Since rank of A is 2, | A | = 0

$\Rightarrow 1(10k-42)-2(20-21)+3(12-3k)=0$

$\Rightarrow 10k-42+2+36-9k=0$

$\Rightarrow k-4=0 \quad \Rightarrow k=4$

11. Prove that the inverse of a non – singular symmetric matrix A is symmetric.

[JNTU 2004]

Solution

A is nonsingular $\Rightarrow |A| \neq 0 \Rightarrow A^{-1}$ exists.

A is symmetric $\Rightarrow$ A' = A ....(1)

$(A^{-1})' = (A')^{-1} = A^{-1}$ from (1) $\Rightarrow A^{-1}$ is symmetric .

12. Prove that the inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal. [JNTU 2005]

Solution

Let A be an orthogonal matrix $\Rightarrow A^1 = A^{-1} \Rightarrow A'.A = A.A' = I$

(i) $(A'A)^{-1} = I \Rightarrow A^{-1}(A')^{-1} = I \Rightarrow A^{-1}(A^{-1})' = I \Rightarrow A^{-1}$ is orthogonal

(ii) $(A'A)' = I \Rightarrow A'(A')' = I \Rightarrow A'$ is orthogonal.

13. Is the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ is orthogonal ? [JNTU 2003]

Solution

Let the given matrix be A ; $AA' = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix} \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

$= \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix}$ (verify) $\neq I \Rightarrow A$ is not orthogonal.

14. Define the inverse of a square matrix and show that every invertible matrix possesses a unique inverse. [JNTU 2003]

Solution

for definition see 1.3.2

Let A be an invertible matrix. Hence A^{-1} exists.

Let, if possible, A possess two inverses say B and C. Then, we have

AB=BA=I and AC=CA=I

Now C=CI=C(AB)= (CA) B=IB =B . Hence the result.

15. If A,B are invertible matrices of the same order, Prove that

(i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $(A')^{-1} = (A^{-1})'$ [JNTU 2002]

Solution

$(i)(AB)(B^{-1}A)^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$.

Similarly

$(B^{-1}A^{-1})(AB)=I$; Therefore $= (AB)^{-1} = B^{-1}A^{-1}$

(ii)

$AA^{-1} = A^{-1}A=I \Rightarrow (A^{-1}A)'=(AA^{-1})'=I'=I.$

$\Rightarrow (A^{-1})'A'=A'(A^{-1})'=I \Rightarrow (A')^{-1}=(A^{-1})'$ (by definition of inverse)

16. Express the matrix A as a sum of symmetric and skew-symmetric matrix where

$$A=\begin{pmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{pmatrix}$$ [JNTU 2002]

Solution

A=B+C , where $B=\frac{1}{2}(A+A')$ is symmetric and $C=\frac{1}{2}(A-A')$

Is skew-symmetric (after 1.1.30 see Ex 3)

$$\therefore B=\frac{1}{2}(A+A')=\frac{1}{2}\left[\begin{pmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{pmatrix}+\begin{pmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{pmatrix}\right]=\begin{bmatrix} 3 & 0 & \frac{11}{2} \\ 0 & 7 & \frac{3}{2} \\ \frac{11}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

which is symmetric

$$C=\frac{1}{2}(A\text{-}A') = \begin{pmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -5/2 \\ -1/2 & 5/2 & 0 \end{pmatrix} \text{ which is skew - symmetric.}$$

$$A = B + C = \begin{bmatrix} 3 & 0 & 11/2 \\ 0 & 7 & 3/2 \\ 11/2 & 3/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -5/2 \\ -1/2 & 5/2 & 0 \end{bmatrix}$$

17. Define ad joint of a matrix and hence find A^{-1} by using ad joint of A where

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution

Let A be a given square matrix . If each element is replaced by its cofactor, the resulting matrix is known as cofactor matrix of A. The transpose of the cofactor matrix is called ad joint of A .

For the given matrix A, $|A| = 1(-12-12)-1(-4-6)+3(-4+6) = -8 \neq 0$

$\therefore$ A is non – singular $\Rightarrow$ A^{-1} exists.

Cofactors of elements of 1^{st} row of A are

$$(-1)^{1+1}\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} ;\ (-1)^{1+2}\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} ;\ (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = -24, 10, 2$$

Similarly finding the other cofactors , we can find that,

$$\text{cofactor matrix} = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix} \text{ (verify)} \Rightarrow \text{Adj A} = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|} adj A = -\frac{1}{8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

18. Find the value of 'x' such that matrix 'A' is singular where

$$A = \begin{pmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -(1+x) \end{pmatrix}$$ [JNTU 2002, 2003]

Solution

A is singular implies $\Rightarrow$ $|A| = 0 \Rightarrow \begin{vmatrix} 3\text{-x} & 2 & 2 \\ 2 & 4\text{-x} & 1 \\ -2 & -4 & -(1+x) \end{vmatrix} = 0$

$$R_3 = R_3 + R_2 \Rightarrow \begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4\text{-x} & 1 \\ 0 & \text{-x} & \text{-x} \end{vmatrix} = 0 \Rightarrow \text{-x}\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4\text{-x} & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0 \quad R_3 = R_3 \times \frac{-1}{x}$$

$$C_3 = C_3 - C_2 \Rightarrow +x\begin{vmatrix} 3-x & 2 & 0 \\ 2 & 4\text{-x} & x\text{-3} \\ 0 & 1 & 0 \end{vmatrix} = 0 \Rightarrow x(x\text{-3})\begin{vmatrix} 3\text{-x} & 3 & 0 \\ 2 & 4\text{-x} & 1 \\ 0 & 1 & 0 \end{vmatrix} \left(C_3 = \frac{1}{x-3}C_3\right)$$

Expanding the det by last column we get, -x(x-3)(3-x)=0 implies x=0 or 3

19. Solve the system of equations by matrix method..

$x_1 + x_2 + x_3 = 2,\ 4x_1 - x_2 + 2x_3 = -6,\ 3x_1 + x_2 + x_3 = -18$ [JNTU 2000S]

Solution

The given system can be written in the form AX = B, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix};\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix};\ B = \begin{bmatrix} 2 \\ -6 \\ -18 \end{bmatrix}$$

$$(A,B) = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 4 & -1 & 2 & -6 \\ 3 & 1 & 1 & -18 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -5 & -2 & -14 \\ 0 & -2 & -2 & -24 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2/5 & 14/5 \\ 0 & 1 & 1 & 12 \end{bmatrix} \begin{matrix} R_2 \to -\frac{1}{5}R_2 \\ R_3 \to -\frac{1}{2}R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3/5 & -4/5 \\ 0 & 1 & 2/5 & 14/5 \\ 0 & 0 & 3/5 & 46/5 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 3/5 & -4/5 \\ 0 & 1 & 2/5 & 14/5 \\ 0 & 0 & 1 & 46/3 \end{bmatrix} R_3 \to (5/3)R_3$$

$$\sim\begin{bmatrix}1 & 0 & 0 & -10\\ 0 & 1 & 0 & -10/3\\ 0 & 0 & 1 & 46/3\end{bmatrix} \begin{matrix}R_1 \to R_1-(3/5)R_2\\ R_2 \to R_2-(2/5)R_3\end{matrix} \quad \therefore x_1=-10, x_2=-(10/3), x_3=46/3$$

Aliter:- work out by taking $X=A^{-1}B$

20. Find the inverse of the matrix $A=\begin{bmatrix}a+ib & c+id\\ -c+id & a-ib\end{bmatrix}$ if $a^2+b^2+c^2+d^2=1$

[JNTU 2003]

Solution

$$|A|=(a+ib)(a-ib)+(c+id)(c-id)=a^2+b^2+c^2+d^2=1$$

Cofactor matrix $=\begin{bmatrix}a-ib & c-id\\ -(c+id) & (a+ib)\end{bmatrix}$; Adj A $=\begin{bmatrix}a-ib & -(c+id)\\ c-id & a+ib\end{bmatrix}$

$$\therefore\ A^{-1}=\frac{1}{|A|}\text{ Adj A }=\begin{bmatrix}a-ib & -(c+id)\\ c-id & a+ib\end{bmatrix}$$

21. Prove that the matrix $A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ is orthogonal. [JNTU 2004]

Solution

$$AA'=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I \quad \text{(verify)}$$

Similarly $A'A=I$;

$\therefore$ A is orthogonal.

22. For what value of '*k*' the matrix $\begin{bmatrix}4 & 4 & -3 & 1\\ 1 & 1 & -1 & 0\\ k & 2 & 2 & 2\\ 9 & 9 & k & 3\end{bmatrix}$ has rank 3.

Solution

Since the rank is 3, the determinant of the matrix is zero.

$$\begin{vmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{vmatrix} = 0 \quad R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2\text{-k} & 2\text{+k} & 2 \\ 0 & 0 & \text{k+9} & 3 \end{vmatrix} = 0 \quad \begin{pmatrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - kR_1 \\ R_4 \to R_4 - 9R_1 \end{pmatrix} \Rightarrow \begin{vmatrix} 0 & 1 & 1 \\ 2\text{-k} & 2\text{+k} & 2 \\ 0 & \text{k+9} & 3 \end{vmatrix} (\text{Expanding by } c_1)$$

$$\Rightarrow (2-k)\begin{vmatrix} 1 & 1 \\ \text{k+9} & 3 \end{vmatrix} = 0 \quad (\text{Expanding by } c_1) \Rightarrow (2-k)(3-k-9) = 0 \Rightarrow k = 2.-6$$

23. Reduce the matrix A to normal form and hence find its rank [JNTU 2003]

$$\text{where A} = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

Solution

$$\text{A} = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix} \quad \begin{matrix} R_3 = R_3 - R_1 \\ R_4 = R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 = R_2 - R_3 \\ R_4 = R_4 - 2R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} c_1 = \frac{1}{2}c_1 \sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_1 = R_1 - R_2 \\ R_3 = R_3 - 2R_2 \end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&4&1\\0&0&0&0\end{bmatrix}\begin{matrix}c_3=c_3-3c_1\\c_4=c_4-4c_1\end{matrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}c_3=\frac{1}{4}c_3$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}c_4=c_4-c_3\text{, which is the normal form }\begin{bmatrix}I_3&0\\0&0\end{bmatrix}$$

$\therefore$ Rank of A = 3

24. By reducing the matrix $\begin{bmatrix}2&3&-1&-1\\1&-1&-2&-4\\3&1&3&-2\\6&3&0&-7\end{bmatrix}$ into normal form, find its rank. [JNTU 2005]

Solution

Applying $R_1 \leftrightarrow R_2$, the given matrix A $\sim\begin{bmatrix}1&-1&-2&-4\\2&3&-1&-1\\3&1&3&-2\\6&3&0&-7\end{bmatrix}$

$$\sim\begin{bmatrix}1&-1&-2&-4\\0&5&3&7\\0&4&9&10\\0&9&12&17\end{bmatrix}\begin{matrix}R_2=R_2-2R_1\\R_3=R_3-3R_1\\R_4=R_4-6R_1\end{matrix}\sim\begin{bmatrix}1&-1&-2&-4\\0&5&3&7\\0&4&9&10\\0&0&0&0\end{bmatrix}R_4=R_4-(R_2+R_3)$$

$$\sim\begin{bmatrix}1&0&0&0\\0&5&3&7\\0&4&9&10\\0&0&0&0\end{bmatrix}\begin{matrix}c_2=c_2+c_1\\c_3=c_3+2c_1\\c_4=c_4+4c_1\end{matrix}\sim\begin{bmatrix}1&0&0&0\\0&1&-6&-3\\0&4&9&10\\0&0&0&0\end{bmatrix}R_2=R_2-R_3$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&-6&-3\\0&0&33&22\\0&0&0&0\end{bmatrix} R_3 = R_3 - 4R_2 \quad \sim\begin{bmatrix}1&0&0&0\\0&1&-6&-3\\0&0&3&2\\0&0&0&0\end{bmatrix} R_3 = R_3 \times \frac{1}{11}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&3&2\\0&0&0&0\end{bmatrix} \begin{matrix} c_3 = c_3 + 6c_2 \\ c_4 = c_4 + 3c_2 \end{matrix} \sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix} \begin{matrix} c_3 = c_3 \times \frac{1}{3} \\ c_4 = c_4 \times \frac{1}{2} \end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix} c_4 = c_4 - c_3 = \begin{bmatrix}I_3&0\\0&0\end{bmatrix}$$, the normal form ; $\therefore$ Rank =3

25. By reducing the matrix $\begin{bmatrix}1&2&3&4\\2&1&4&3\\3&0&5&-10\end{bmatrix}$ into normal form , find its rank.

[JNTU 2002]

Solution

$$A = \begin{bmatrix}1&2&3&4\\2&1&4&3\\3&0&5&-10\end{bmatrix} \sim \begin{bmatrix}1&2&3&4\\0&-3&-2&-5\\0&-6&-4&-22\end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&-3&-2&-5\\0&-6&-4&-22\end{bmatrix} \begin{matrix} c_2 = c_2 - 2c_1 \\ c_3 = c_3 - 3c_1 \\ c_4 = c_4 - 4c_1 \end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&-3&-2&-5\\0&0&0&-12\end{bmatrix} R_3 = R_3 - 2R_2 \sim \begin{bmatrix}1&0&0&0\\0&1&1&5\\0&0&0&12\end{bmatrix} \begin{matrix} c_2 = -\frac{1}{3}c_2 \\ c_3 = -\frac{1}{2}c_3 \\ c_4 = -c_4 \end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&12\end{bmatrix}\begin{matrix}c_3=c_3-c_2\\c_4=c_4-5c_3\end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}c_4=c_4\times\frac{1}{12}\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}c_4\leftrightarrow c_3=\begin{bmatrix}I_3&0\end{bmatrix}$$

which is the normal form . Hence rank of A $=3$.

26. Reduce A to canonical form and find its rank , if A $=\begin{bmatrix}1&2&3&0\\2&4&3&2\\3&2&1&3\\6&8&7&5\end{bmatrix}$ [JNTU 2002]

Solution

Applying $R_2=R_2-2R_1, R_3=R_3-3R_1, R_4=R_4-6R_1$ to A , we get,

$$A\sim\begin{bmatrix}1&2&3&0\\0&0&-3&2\\0&-4&-8&3\\0&-4&-11&5\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&0&-3&2\\0&-4&-8&3\\0&-4&-11&5\end{bmatrix}\begin{matrix}c_2=c_2-2c_1\\c_3=c_3-3c_1\end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&0&-3&2\\0&-4&-8&3\\0&0&-3&2\end{bmatrix}R_4=R_4-R_3\sim\begin{bmatrix}1&0&0&0\\0&0&-3&2\\0&-4&-8&3\\0&0&0&0\end{bmatrix}R_4=R_4-R_2$$

$$\sim\begin{bmatrix}1&0&0&0\\0&0&-3&2\\0&1&-8&3\\0&0&0&0\end{bmatrix}c_2=-\frac{1}{4}c_2\sim\begin{bmatrix}1&0&0&0\\0&1&-8&3\\0&0&-3&2\\0&0&0&0\end{bmatrix}R_2\leftrightarrow R_3$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-3&2\\0&0&0&0\end{bmatrix}\begin{matrix}c_3=c_3+8c_2\\c_4=c_4+3c_2\end{matrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}\begin{matrix}c_3=-\frac{1}{3}c_3\\c_4=\frac{1}{2}c_4\end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix} \quad c_4 = c_4 - c_3 = \begin{bmatrix}I_3 & 0\\ 0 & 0\end{bmatrix}$$ which is the canonical form.

$\therefore$ Rank of A = 3

27. Find the rank of the matrix A by reducing it to the normal form

where $A = \begin{bmatrix}1&1&1&1\\1&2&3&-4\\2&3&5&-5\\3&-4&-5&8\end{bmatrix}$ [JNTU 2005]

Solution

$$A \sim \begin{bmatrix}1&1&1&1\\0&1&2&-5\\0&1&3&-7\\0&-7&-8&5\end{bmatrix} \begin{matrix}R_2 = R_2 - R_1\\R_3 = R_3 - 2R_1\\R_4 = R_4 - 3R_1\end{matrix} \sim \begin{bmatrix}1&0&0&0\\0&1&2&-5\\0&1&3&-7\\0&-7&-8&5\end{bmatrix} \begin{matrix}c_2 = c_2 - c_1\\c_3 = c_3 - c_1\\c_4 = c_4 - c_1\end{matrix}$$

$$\sim \begin{bmatrix}1&0&0&0\\0&1&2&-5\\0&0&1&-2\\0&0&6&-30\end{bmatrix} \begin{matrix}R_3 = R_3 - R_2\\R_4 = R_4 + 7R_2\end{matrix} \sim \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-2\\0&0&6&-30\end{bmatrix} \begin{matrix}C_3 = C_3 - 2C_2\\C_4 = C_4 + 5C_2\end{matrix}$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-2\\0&0&0&-18\end{bmatrix} \quad R_4 = R_4 - 6R_3$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-18\end{bmatrix} C_4 = C_4 + 2C_3 \sim \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} R_4 = \frac{-1}{18}R_4$$

$\sim I_4$. Therefore Rank of A= 4

28. Find nonsingular matrices P and Q such that the normal form of A is

PAQ where $A=\begin{bmatrix}1 & 3 & 6 & -1\\ 1 & 4 & 5 & 1\\ 1 & 5 & 4 & 3\end{bmatrix}$ [JNTU 2005]

Solution Since A is of order 3×4 ,write $A = I_3 A I_4$; i.e;

$$\begin{bmatrix}1 & 3 & 6 & -1\\ 1 & 4 & 5 & 1\\ 1 & 5 & 4 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

We apply elementary row and corresponding column transformations

On L.H.S. We apply row transformations to prefactor $(I_3)$and column transformations to postfactor (I_4) of R.H.S.

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - R_1$ to L.H.S and I_3, we get

$$\begin{bmatrix}1 & 6 & 3 & -1\\ 0 & 1 & -1 & 2\\ 0 & 2 & -2 & 4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Applying $c_2 \to c_2 - 3c_1$, $c_3 \to c_3 - 6c_1$,$c_4 \to c_4 + c_1$ (to L.H.S and I_4) we get

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -1 & 2\\ 0 & 2 & -2 & 4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -3 & -6 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Apply $R_3 \to R_3 - 2R_2$, to get $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -2 & 1\end{bmatrix} A \begin{bmatrix}1 & -3 & -6 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

Apply $c_3 \to c_3 + c_2$, $c_4 \to c_4 - 2c_2$, to get

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\1&-2&1\end{bmatrix}A\begin{bmatrix}1&-3&-9&7\\0&1&1&-2\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}I_2&0\\0&0\end{bmatrix}=PAQ,\ \text{where } P=\begin{bmatrix}1&0&0\\-1&1&0\\1&-2&1\end{bmatrix},\ Q=\begin{bmatrix}1&-3&-9&7\\0&1&1&-2\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Therefore Rank of A = 2.

29. If $\mathbf{A}=\begin{bmatrix}2&1&-3&-6\\2&-3&1&2\\1&1&1&2\end{bmatrix}$,find non-singular matrices P and Q such that PAQ is in normal form. [JNTU 2002]

Solution

A is of order 3×4. Therefore write $A=I_3 A\, I_4$

$$\text{i.e} \sim\begin{bmatrix}2&1&-3&-6\\2&-3&1&2\\1&1&1&2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Apply elementary row operations on L.H.S and on the pre-factor I_3 of R.H.S and elementary column transformations on L.H.S and on the post-factor I_4 of R.H.S..

$$R_1\leftrightarrow R_3\Rightarrow\begin{bmatrix}1&1&1&2\\2&-3&1&2\\2&1&-3&-6\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$R_2=R_2-2R_1, R_3=R_3-2R_1\ \text{gives}\ \begin{bmatrix}1&1&1&2\\0&-5&-1&-2\\0&-1&-5&-10\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-2\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$c_2=c_2-c_1$,$c_3 = c_3- c_1$, $c_4= c_4-2\, c_3$ gives

$$\begin{bmatrix}1&0&0&0\\0&-5&-1&0\\0&-1&-5&0\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-2\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&-1&-1&0\\0&1&0&0\\0&0&1&-2\\0&0&0&1\end{bmatrix}$$

Multiplying 2^{nd} row and 3^{rd} row each with -1, and then

interchanging 2^{nd} and 3^{rd} rows, we get,

$$\begin{bmatrix}1&0&0&0\\0&1&5&0\\0&5&1&0\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&0&2\\0&-1&2\end{bmatrix}A\begin{bmatrix}1&-1&-1&0\\0&1&0&0\\0&0&1&-2\\0&0&0&1\end{bmatrix}$$

$R_3 =R_3-5R_2$ gives $\begin{bmatrix}1&0&0&0\\0&1&5&0\\0&0&-24&0\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&0&2\\5&-1&-8\end{bmatrix}A\begin{bmatrix}1&-1&-1&0\\0&1&0&0\\0&0&1&-2\\0&0&0&1\end{bmatrix}$

$c_3=c_3-5c_2$ gives $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-24&0\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&0&2\\5&-1&-8\end{bmatrix}A\begin{bmatrix}1&-1&4&0\\0&1&-5&0\\0&0&1&-2\\0&0&0&1\end{bmatrix}$

$R_3 =-\frac{R_3}{24}$ gives $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&0&2\\\frac{-5}{24}&\frac{1}{24}&\frac{1}{3}\end{bmatrix}A\begin{bmatrix}1&-1&4&0\\0&1&-5&0\\0&0&1&-2\\0&0&0&1\end{bmatrix}$

$[I_3 \quad 0]= PAQ$,where

$$P=\begin{bmatrix}0&0&1\\-1&0&2\\\frac{-5}{24}&\frac{1}{24}&\frac{1}{3}\end{bmatrix},Q=\begin{bmatrix}1&-1&4&0\\0&1&-5&0\\0&0&1&-2\\0&0&0&1\end{bmatrix}$$

P and Q are non-singular and Rank of A =3

30. Find the non-singular matrices P and Q such that PAQ is in the normal form of the matrix and find the rank of the matrix,

$$A=\begin{bmatrix}1 & 2 & 3 & -2\\ 2 & -2 & 1 & 3\\ 3 & 0 & 4 & 1\end{bmatrix}$$ [JNTU 2005S]

Solution

A is of order 3x4. Therefore write $A= I_3\ A\ I_4$,i.e;

$$\begin{bmatrix}1 & 2 & 3 & -2\\ 2 & -2 & 1 & 3\\ 3 & 0 & 4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Let us apply elementary row transformations on the L.H.S and the Prefactor I_3 of R.H.S. Also apply column transformations to the L.H.S and the post factor I_4 of R.H.S.

$R_2 \rightarrow R_2 - 2R_1,\ R_3 \rightarrow R_3\text{-}3R_1$ gives $\begin{bmatrix}1 & 2 & 3 & -2\\ 0 & -6 & -5 & 7\\ 0 & -6 & -5 & 7\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

$R_3 \rightarrow R_3\text{-}R_2$ gives $\begin{bmatrix}1 & 2 & 3 & -2\\ 0 & -6 & -5 & 7\\ 0 & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & -1 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

$c_2 \rightarrow c_2 - 2c_1, c_3 \rightarrow c_3\text{-}3c_1, c_4 \rightarrow c_4 + 2c_1$ gives

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -6 & -5 & 7\\ 0 & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & -1 & 1\end{bmatrix}A\begin{bmatrix}1 & -2 & -3 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$ .Now $c_2 \rightarrow \frac{-1}{6}c_2$ gives

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & -5 & 7\\0 & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\-1 & -1 & 1\end{bmatrix}A\begin{bmatrix}1 & \frac{1}{3} & -3 & 2\\0 & \frac{-1}{6} & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$c_3 \to c_3+5c_2, c_4 \to c_4-7c_2$ gives

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\-1 & -1 & 1\end{bmatrix}A\begin{bmatrix}1 & 1/3 & -4/3 & -1/3\\0 & -1/6 & -5/6 & 7/6\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

i..e ; $\begin{bmatrix}I_2 & 0\\0 & 0\end{bmatrix}$(normal form) = PAQ ,where $P=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\-1 & -1 & 1\end{bmatrix}$

and $Q=\begin{bmatrix}1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3}\\0 & \frac{-1}{6} & -\frac{5}{6} & \frac{7}{6}\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$; P,Q are non-singular and

Rank of A = 2.

31. Determine the non-singular matrices P and Q such that PAQ is in the normal form for A.

Hence find the rank of the matrix A where $A=\begin{pmatrix}2 & -1 & 3\\1 & 1 & 1\\1 & -1 & 1\end{pmatrix}$ [JNTU 2004]

Solution

A is a 3 X 3 matrix; $\therefore$ write $A = I_3\ A\ I_3$

i.e. $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Subject the L H S and pre-factor on R H S to elementary row transformations and apply column transformations to L H S and post-factor of R H S .

$$R_2 \leftrightarrow R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2=R_2-2R_1;\ R_3-R_1;\ \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2= C_2-C_1;\ C_3 = C_3-C_1;\ \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2=R_2\times-\frac{1}{3};\ R_3=R_3\times-\frac{1}{2};\ \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & 1/2 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3=R_3-R_2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/3 & 2/3 & 0 \\ 1/3 & -1/6 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3=C_3+1/3C_2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/3 & 2/3 & 0 \\ 1/3 & -1/6 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3= 3C_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/3 & 2/3 & 0 \\ 1/3 & -1/6 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$I_3 = PAQ$, where $P = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{2} \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

P, Q are non –singular (verify) , Rank of A = 3

32. Compute the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$ by using elementary operations.

[JNTU 2004, 2005]

Solution

Write $A = I_4 A$, and apply elementary row operations on the L.H.S as well as I_4

till we get the equation of the form $I_4 = B\,A$ where $B = A^{-1}$

Now, $A = I_4 A \Rightarrow \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$

$R_2 \leftrightarrow R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$

$R_3 = R_3 - 2R_1$; $R_4 = R_4 - 2R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$

$$R_1 = R_1 - R_2 \; ; \; R_4 = R_4 - R_2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} A$$

$$R_3 = R_3 \times -\frac{1}{2} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & +1 & -1/2 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} A$$

$$R_2 = R_2 - 2R_3 \; ; \; R_4 = R_4 + 3R_3 ; \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & -1/2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ -1 & 1 & -3/2 & 1 \end{bmatrix} A$$

$$R_4 = -2R_4 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

$$\left.\begin{array}{l} R_1 = R_1 - R_4 ; \\ R_2 = R_2 + R_4 ; \\ R_3 = R_3 - (3/2) R_4 ; \end{array}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$$

33. Find the inverse of the matrix A using elementary operations where

$$A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

[JNTU 2004]

Solution Let $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$ .....(1)

Apply elementary row operations on L.H.S. and the prefactor of R.H.S. of (1) till we get

$$I_4 = B\,A \text{ where } B = A^{-1}$$

$$R_1 = -1 \times R_1 \Rightarrow \begin{bmatrix} 1 & 3 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$\left.\begin{matrix} R_2 = R_2 - R_1\ ;\ R_3 = R_3\text{-}2R_1; \\ R_4 = R_4 + R_1 \end{matrix}\right\} \Rightarrow \begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} A$$

$$R_2 = R_2 \times -\frac{1}{2} \Rightarrow \begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & -1 & 1/2 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

$$\left.\begin{matrix} R_1 = R_1\text{-}3R_2 \\ R_3 = R_3 + 11R_2 \\ R_4 = R_4\text{-}4R_2 \end{matrix}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & -3 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ -7/2 & -11/2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$$R_3 = R_3 \times -\frac{1}{3} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & -1/6 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ 7/6 & 11/6 & -1/3 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$$R_2 = R_2 + R_3 \; ; \; R_4 = R_4 - R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & -1/6 \\ 0 & 0 & 0 & 1/6 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 & 0 & 0 \\ 2/3 & 4/3 & -1/3 & 0 \\ 7/6 & 11/6 & -1/3 & 0 \\ -1/6 & 1/6 & 1/3 & 1 \end{bmatrix} A$$

$$R_4 = R_4 \times 6 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & -1/6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 & 0 & 0 \\ 2/3 & 4/3 & -1/3 & 0 \\ 7/6 & 11/6 & -1/3 & 0 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$$\left.\begin{aligned} R_1 &= R_1 + \frac{1}{2} R_4 \\ R_2 &= R_2 - \frac{1}{3} R_4 \\ R_3 &= R_3 + \frac{1}{6} R_4 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$$\therefore \; A^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

34. Find the inverse of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ by elementary row operations.

[JNTU 2006]

Solution

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{bmatrix} \begin{pmatrix} R_2 = R_2 - R_1 \\ R_3 = R_3 + 2R_1 \end{pmatrix}$$

$$\sim\begin{bmatrix}1&1&3&1&0&0\\0&1&-3&-1/2&1/2&0\\0&-2&2&2&0&1\end{bmatrix}\left(R_2=R_2\times\frac{1}{2}\right)\sim\begin{bmatrix}1&0&6&3/2&-1/2&0\\0&1&-3&-1/2&1/2&0\\0&0&-4&1&1&1\end{bmatrix}\begin{matrix}R_1=R_1-R_2\\R_3=R_3+2R_2\end{matrix}$$

$$\sim\begin{bmatrix}1&0&6&3/2&-1/2&0\\0&1&-3&-1/2&1/2&0\\0&0&1&-1/4&-1/4&-1/4\end{bmatrix}\left(R_3=R_3\times-\frac{1}{4}\right)\sim\begin{bmatrix}1&0&0&3&1&3/2\\0&1&0&-5/4&-1/4&-3/4\\0&0&1&-1/4&-1/4&-1/4\end{bmatrix}\begin{matrix}R_1=R_1-6R_3\\R_2=R_2+3R_3\end{matrix}$$

$$\therefore \text{ Required inverse} = \begin{bmatrix}3&1&3/2\\-5/4&-1/4&-3/4\\-1/4&-1/4&-1/4\end{bmatrix}$$

Aliter: Try the problem as in the above problem.

35. Find the values of 'a' and 'b' for which the equations $x+y+z=3$; $x+2y+2z=6$; $x+ay+3z=b$ have (i) no solution (ii) a unique solution (iii) infinite number of solutions. [JNTU 2001]

Solution The given system, if written in matrix form AX = B, we get,

$$A=\begin{bmatrix}1&1&1\\1&2&2\\1&a&3\end{bmatrix};\ X=\begin{bmatrix}x\\y\\z\end{bmatrix};\ B=\begin{bmatrix}3\\6\\b\end{bmatrix};$$

$$\text{Augmented matrix } [A, B] = \begin{bmatrix}1&1&1&3\\1&2&2&6\\1&a&3&b\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1&3\\0&1&1&3\\0&(a-1)&2&(b-3)\end{bmatrix}\begin{pmatrix}R_2=R_2-R_1\\R_3=R_3-R_1\end{pmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&1&3\\0&a-3&0&b-9\end{bmatrix}\begin{pmatrix}R_1=R_1-R_2\\R_3=R_3-2R_2\end{pmatrix}$$

(i) If $a=3$; $b\neq 9$; Rank of A = 2 ; Rank of (A , B) = 3 $\neq$ rank of A.

$\therefore$ The system has no solution.

(ii) If $a\neq 3$, Rank of A = Rank of (A , B) = 3 ; i.e., the system has a unique solution.

(iii) If $a=3$ and $b=9$, Rank of A = Rank of (A , B) = 2 < 3 (number of variables).

Then the system will have an infinite number of solutions.

36. Find whether the following equations are consistent. If so solve them;

[JNTU 2005, 2005 S]

$$x+y+2z=4 \;;\; 2x-y+3z=9 \;;\; 3x-y-z=2$$

Solution

The given system is $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$ is of the form AX = B

$\therefore$ The augmented matrix (A, B) = $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix} \begin{pmatrix} R_2=R_2-2R_1 \\ R_3=R_3-3R_1 \end{pmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 4 & 7 & 10 \end{bmatrix} \begin{pmatrix} R_2=-\frac{1}{3}R_2 \\ R_4=-R_4 \end{pmatrix} \sim \begin{bmatrix} 1 & 0 & 5/3 & 13/3 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 17/3 & 34/3 \end{bmatrix} \begin{pmatrix} R_1=R_1-R_2 \\ R_3=R_3-4R_2 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 5/3 & 13/3 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \left(R_3=R_3\times\frac{3}{17}\right) \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{pmatrix} R_1=R_1-\frac{5}{3}R_3 \\ R_2=R_2-\frac{1}{3}R_3 \end{pmatrix}$$

Which shows that Rank of (A, B) = Rank of A = 3 (no. of unknowns).

$\therefore$ The system is consistent and has a unique solution. From the final matrix, we get,

$x=1;\; y=-1;\; z=2$

37. Find the value of λ for which the system of equations

$3x-y+4z=3;\; x+2y-3z=-2;\; 6x+5y+\lambda z=-3$ will have infinite number of solutions and solve them with that λ value.

[JNTU 2005 S]

Solution

If the given system is put in the matrix form AX = B,

$$A=\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix};\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix};\ B=\begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

Augmented matrix $[A,B]=\begin{bmatrix} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & -2 \\ 3 & -1 & 4 & 3 \\ 6 & 5 & \lambda & -3 \end{bmatrix} (R_1 \leftrightarrow R_2)$

$$\sim \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & -7 & \lambda+18 & 9 \end{bmatrix} \begin{pmatrix} R_2=R_2-3R_1 \\ R_3=R_3-6R_1 \end{pmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & \lambda+5 & 0 \end{bmatrix} (R_3=R_3-R_2)$$

If $\lambda=-5$, $R(A)=R(A,B)=2<3$ (number of unknowns) and the system will have an infinite number of solutions involving $3-2=1$ arbitrary constant.

If λ = -5 , the final matrix becomes $\begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow\ x+2y-3z=-2;\ -7y+13z=9$; Taking z = c , we get ,

$$y=\frac{1}{7}(13c-9);x=\frac{-2}{7}(13c-9)+3c-2=\frac{-5c}{7}+\frac{4}{7}$$

$\therefore$ The solution of the system is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} \frac{1}{7}(4-5c) \\ \frac{1}{7}(13c-9) \\ c \end{bmatrix}$

38. Find whether the following set of equations are consistent. If so, solve them:

$x_1+x_2+x_3+x_4=0; \quad x_1+x_2+x_3-x_4=4;$

$x_1+x_2-x_3+x_4=-4; \quad x_1-x_2+x_3+x_4=2;$

[JNTU 2005, 2006]

Solution

The augmented matrix of the given system AX = B, is

$$[A,B]=\begin{bmatrix}1&1&1&1&0\\1&1&1&-1&4\\1&1&-1&1&-4\\1&-1&1&1&2\end{bmatrix}\sim\begin{bmatrix}1&1&1&1&0\\0&0&0&-2&4\\0&0&-2&0&-4\\0&-2&0&0&2\end{bmatrix}\begin{pmatrix}R_2=R_2-R_1\\R_3=R_3-R_1\\R_4=R_4-R_1\end{pmatrix}$$

which is in echolon form which has 4 non-zero rows.

$\therefore$ $R(A)=R(A/B)=4$ $\quad\therefore$ The system is consistent and equivalent to

$x_1+x_2+x_3+x_4=0\ ;\ -2x_4=4\ ;\ -2x_3=-4\ ;\ -2x_2=2$

$\therefore\ x_4=-2\ ;\ x_3=2\ ;\ x_2=-1\ ;\ x_1=1$. is the solution.

39. Show that the equations $x+2y-z=3\ ;\ 3x-y+2z=1\ ;\ 2x-2y+3z=2\ ;$

$x-y+z=-1$ are consistent and solve them.

Solution

If AX = B is the given system , $A=\begin{bmatrix}1&2&-1\\3&-1&2\\2&-2&3\\1&-1&1\end{bmatrix}$; $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$; $B=\begin{bmatrix}3\\1\\2\\-1\end{bmatrix}$

$\therefore$ Augmented matrix $[A,B]=\begin{bmatrix}1&2&-1&3\\3&-1&2&1\\2&-2&3&2\\1&-1&1&-1\end{bmatrix}\sim\begin{bmatrix}1&2&-1&3\\0&-7&5&-8\\0&-6&5&-4\\0&-3&2&-4\end{bmatrix}\begin{pmatrix}R_2=R_2-3R_1\\R_3=R_3-2R_1\\R_4=R_4-R_1\end{pmatrix}$

$$\sim\begin{bmatrix}1 & 2 & -1 & 3\\ 0 & 1 & -5/7 & 8/7\\ 0 & -6 & +5 & -4\\ 0 & -3 & 2 & -4\end{bmatrix}\left(R_2=\frac{-1}{7}\times R_2\right)\sim\begin{bmatrix}1 & 2 & -1 & 3\\ 0 & 1 & -5/7 & 8/7\\ 0 & 0 & 5/7 & 20/7\\ 0 & 0 & -1/7 & -4/7\end{bmatrix}\begin{pmatrix}R_3=R_3+6R_2\\ R_4=R_4+3R_2\end{pmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 3\\ 0 & 1 & -5/7 & 8/7\\ 0 & 0 & 1 & 4\\ 0 & 0 & 1 & 4\end{bmatrix}\begin{pmatrix}R_3=R_3\times\frac{7}{5}\\ R_4=R_4\times -7\end{pmatrix}\sim\begin{bmatrix}1 & 2 & -1 & 3\\ 0 & 1 & -5/7 & 8/7\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 0\end{bmatrix}(R_4=R_4-R_3)$$

Which is in echelon form with 3 non – zero rows.

$\therefore$ R(A) = R(A/B) = 3 (number of unknown) . $\therefore$ the system consistent and has unique solution.

The system is equivatent to $x+2y-z=3$; $y-\frac{5}{7}z=\frac{8}{7}$; $z=4$

$\therefore$ we have $z=4$; $y=4$; $x=-1$;

40. Solve : $x+y+z=6$; $x-y+2z=5$; $2x-2y+3z=7$

[JNTU 2000]

Solution

$$AX=B \Rightarrow A=\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 2\\ 2 & -2 & 3\end{bmatrix}; X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}; B=\begin{bmatrix}6\\ 5\\ 7\end{bmatrix}$$

Augmented Matrix $[A, B]=\begin{bmatrix}1 & 1 & 1 & 6\\ 1 & -1 & 2 & 5\\ 2 & -2 & 3 & 7\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & -2 & 1 & -1\\ 0 & -4 & 1 & -5\end{bmatrix}\begin{pmatrix}R_2=R_2-R_1\\ R_3=R_3-2R_1\end{pmatrix}$

$\sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & -2 & 1 & -1\\ 0 & 0 & -1 & -3\end{bmatrix}(R_3=R_3-2R_2)$. This is echelon form which has 3 non – zero rows .

$\therefore$ Rank of (A, B) = R (A) =3 ; $\therefore$ The system has unique solution and is equivalent to

$x+y+z=6$; $-2y+z=-1$; $z=3 \Rightarrow z=3, y=2, x=1$.

41. Test for consistency and solve : $2x+3y+7z=5$; $3x+y-3z=12$;
$2x+19y-47z=32$ [JNTU 2003 S]

Solution

$$[A, B]=\begin{bmatrix} 2 & 3 & 7 & 5 \\ 3 & 1 & -3 & 12 \\ 2 & 19 & -47 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/2 & 7/2 & 5/2 \\ 3 & 1 & -3 & 12 \\ 2 & 19 & -47 & 32 \end{bmatrix}\left(R_1=R_1\times\frac{1}{2}\right)$$

$$\sim \begin{bmatrix} 1 & 3/2 & 7/2 & 5/2 \\ 0 & -7/2 & -27/2 & 9/2 \\ 0 & 16 & -54 & 27 \end{bmatrix}\begin{pmatrix} R_2=R_2-3R_1 \\ R_3=R_3-2R_1 \end{pmatrix} \sim \begin{bmatrix} 1 & 3/2 & 7/2 & 5/2 \\ 0 & 1 & 27/7 & -9/7 \\ 0 & 16 & -54 & 27 \end{bmatrix}\left(R_2=R_2\times\frac{-2}{7}\right)$$

$$\sim \begin{bmatrix} 1 & 3/2 & 7/2 & 5/2 \\ 0 & 1 & 27/7 & -9/7 \\ 0 & 0 & -810/7 & 333/7 \end{bmatrix}(R_3 = R_3 - 16R_2) \Rightarrow$$ system is consistent since Rank of [A, B] = Rank of A = 3

Further, $x+(3/2)y+(7/2)z=5/2$; $y+(27/7)z=-9/7$; $(-810/7)z=333/7$

$$\therefore\ z=\frac{-333}{810}=\frac{-37}{90}\ ;\ y=\frac{-9}{7}+\frac{27}{7}\times\frac{-37}{90}=\frac{3}{10}\ ;\ x=\frac{5}{2}-\frac{3}{2}.\frac{3}{10}+\frac{7}{2}.\frac{37}{90}=\frac{157}{45}$$

$$\therefore \text{ The solution is } \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{90}\begin{bmatrix} 314 \\ 27 \\ -37 \end{bmatrix}$$

42. Find the values of a and b for which the equations $x+ay+z=3$; $x+2y+2z=b$; $x+5y+3z=9$ are consistent. When all these equations have a unique solution?

[JNTU 2004]

Solution

The augmented matrix of the given system is

$$[A,B]=\begin{bmatrix}1 & a & 1 & 3\\ 1 & 2 & 2 & b\\ 1 & 5 & 3 & 9\end{bmatrix}\sim\begin{bmatrix}1 & a & 1 & 3\\ 0 & (2\text{-}a) & 1 & (b-3)\\ 0 & (5\text{-}a) & 2 & 6\end{bmatrix}\begin{pmatrix}R_2\text{=}R_2\text{-}R_1\\ R_3\text{=}R_3\text{-}R_1\end{pmatrix}$$

$$\sim\begin{bmatrix}1 & a & 1 & 3\\ 0 & (2\text{-}a) & 1 & (b\text{-}3)\\ 0 & 3 & 1 & (9\text{-}b)\end{bmatrix}(R_3\text{=}R_3\text{-}R_2)\sim\begin{bmatrix}1 & a & 1 & 3\\ 0 & (2-a) & 1 & (b-3)\\ 0 & (1+a) & 0 & (12-2b)\end{bmatrix}(R_3 = R_3 - R_2)$$

Case (i) If $a=-1$, $b=6$, $R(A)=R(A,B)=2<3$ (no. of unknowns)[$\because$ there will be 2 non zero rows]

Then the system is consistent and has infinite solutions .

Case(ii) If $a\neq-1$, $R(A) = R(A,B) = 3 \Rightarrow$ system has unique solution.

43. Solve the system of equations $x+3y-2z=0$; $2x-y+4z=0$; $x-11y+14z=0$ [JNTU 2002]

Solution

The given system can be written as AX = 0 where $A=\begin{bmatrix}1 & 3 & \text{-}2\\ 2 & \text{-}1 & 4\\ 1 & \text{-}11 & 14\end{bmatrix}$

$$X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\ 0\text{=}\begin{bmatrix}0\\ 0\\ 0\end{bmatrix};$$

$$A=\begin{bmatrix}1 & 3 & \text{-}2\\ 2 & \text{-}1 & 4\\ 1 & \text{-}11 & 14\end{bmatrix}\sim\begin{bmatrix}1 & 3 & \text{-}2\\ 0 & \text{-}7 & 8\\ 0 & \text{-}14 & 16\end{bmatrix}\begin{pmatrix}R_2\text{=}R_2\text{-}2R_1\\ R_3\text{=}R_3\text{-}R_1\end{pmatrix}$$

$$\sim\begin{bmatrix}1 & 3 & \text{-}2\\ 0 & \text{-}7 & 8\\ 0 & 0 & 0\end{bmatrix}(R_3\text{=}R_3\text{-}2R_2),$$ which is in echelon form with 2 non – zero rows.

$\therefore$ Rank of A = 2 < 3 (number of unknowns).

$\therefore$ The system has infinite number of solutions which involve 3 – 2 = 1 arbitrary constant. The final equivalent matrix of A $\Rightarrow$ $x+3y-2z=0$; $-7y+8z=0$;

$\therefore$ Taking $z=c$, we get $y=\frac{8}{7}c$, $x=-3y+2z=\frac{-24}{7}c+2c \Rightarrow x=\frac{10\text{ c}}{7}$

$\therefore$ $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{c}{7}\begin{bmatrix} -10 \\ 8 \\ 7 \end{bmatrix}$ where c is arbitrary constant.

44. Find the value of λ for which the equations

$(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$

$(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$

$2x+(3\lambda+1)y+3(\lambda-1)z=0$; are consistent and find the ratio of $x:y:z$ when λ has the smallest of these values. What happens when λ has the greater of these values.

[JNTU 2004S]

Solution

The given system is of the form AX = 0

$\therefore$ It is consistent (having non – trivial solution) if | A | = 0 .

i.e., $\begin{vmatrix} \lambda\text{-}1 & 3\lambda\text{+}1 & 2\lambda \\ \lambda\text{-}1 & 4\lambda\text{-}2 & \lambda\text{+}3 \\ 2 & 3\lambda\text{+}1 & 3\lambda\text{-}3 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} \lambda-1 & 3\lambda\text{+}1 & 2\lambda \\ 0 & \lambda-3 & \text{-}\lambda\text{+}3 \\ \text{-}\lambda+3 & 0 & \lambda\text{-}3 \end{vmatrix}=0\begin{pmatrix} R_2\text{-}R_1 \\ R_3\text{-}R_1 \end{pmatrix}$

$\Rightarrow(\lambda-3)^2\begin{vmatrix} \lambda-1 & 3\lambda\text{+}1 & 2\lambda \\ 0 & 1 & \text{-}1 \\ \text{-}1 & 0 & 1 \end{vmatrix}=0$ (Taking $(\lambda-3)$ common in R_2 and R_3)

$\Rightarrow(\lambda-3)^2\left[1(\lambda-1+2\lambda)+(0+3\lambda+1)\right]=0$ (Expanding by R_2)

$\Rightarrow (\lambda-3)^2(6\lambda)=0 \Rightarrow \lambda=0$ (or) 3.

Case (i) : $\lambda=0$ gives $A=\begin{bmatrix} -1 & 1 & 0 \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{bmatrix}$

$$\sim\begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}\begin{pmatrix} R_2=R_2-R_1 \\ R_3=R_3+2R_2 \end{pmatrix}\sim\begin{bmatrix} -1 & +1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}[R_3=R_3+R_2]$$

$\Rightarrow -x+y=0$; $-3y+3z=0 \quad \Rightarrow \quad x=y=z$

Case(ii): $\lambda=3 \Rightarrow A=\begin{bmatrix} 2 & 10 & 6 \\ 2 & 10 & 6 \\ 2 & 10 & 6 \end{bmatrix} \Rightarrow x+5y+z=0$

$\therefore$ Taking $y=\alpha,\ z=\beta,$ we have $x=-5\alpha-\beta,$ where α,β are arbitrary constants.

45. Solve completely the system of equations:

$3x+4y-z-6w=0;\qquad 2x+3y+2z-3w=0;\qquad 2x+y-14z-9w=0;$
$x+3y+13z+3w=0$

Solution

If the system is AX = 0 , $A=\begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}\sim\begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix}(R_1\leftrightarrow R_4)$

$$\sim\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix}\begin{pmatrix} R_2=R_2-2R_1 \\ R_3=R_3-2R_1 \\ R_4=R_4-3R_1 \end{pmatrix}\sim\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & 0 & 0 & 0 \end{bmatrix}(R_4=R_4-R_3)$$

$$\sim\begin{bmatrix}1 & 3 & 13 & 3\\0 & 1 & 8 & 3\\0 & 1 & 8 & 3\\0 & 0 & 0 & 0\end{bmatrix}\begin{pmatrix}R_2=R_2\times\frac{1}{3}\\R_3=R_3\times\frac{1}{5}\end{pmatrix}\sim\begin{bmatrix}1 & 3 & 13 & 3\\0 & 1 & 8 & 3\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}(R_3=R_3\text{-}R_2)$$

Rank of A = 2 < 4 (number of unknowns)

$\therefore$ The solution contains 2 arbitrary constants.

The system becomes $x+3y+13z+3w=0$; $y+8z+3w=0$

Let $z=c_1$; $w=c_2$; $\therefore$ $y=-8c_1-3c_2$ and

$x=-3y-13z-3w=24c_1+9c_2-13c_1-3c_2=11c_1+6c_2$

$\therefore$ $X=\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}11c_1+6c_2\\-8c_1-3c_2\\c_1\\c_2\end{bmatrix}=c_1\begin{bmatrix}11\\-8\\1\\0\end{bmatrix}+c_2\begin{bmatrix}6\\-3\\0\\1\end{bmatrix}$ in the solution where c_1, c_2 are arbitrary constants.

46. Show that the system of equations $2x_1-2x_2+x_3=\lambda x_1$; $2x_1-3x_2+2x_3=\lambda x_2$; $-x_1+2x_2=\lambda x_3$; can possess a non – trivial solution only if $\lambda=1,\ \lambda=-3$; obtain the general solution in each case. [JNTU 2003]

Solution

The given system is $\begin{bmatrix}(2-\lambda) & -2 & 1\\2 & -(\lambda+3) & 2\\-1 & 2 & -\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0(AX=0)$

The system possess a non – trivial solution iff coefficient matrix A,is non – singular.

i.e.; $\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -\lambda-3 & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1-\lambda \\ 0 & -(\lambda+1) & 2-2\lambda \\ -1 & 2 & -\lambda \end{vmatrix} = 0 \begin{pmatrix} R_1=R_1+R_3 \\ R_2=R_2+2R_3 \end{pmatrix}$

$$=(1-\lambda)^2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0 \quad (\text{Taking } (1\text{-}\lambda) \text{common in } R_1 \text{and } R_2)$$

$$\Rightarrow (1-\lambda)^2\left[1(-\lambda-4)+1(1)\right]=0 \quad (\text{Expanding by } R_1)$$

$$\Rightarrow -(1-\lambda)^2(\lambda+3)=0 \Rightarrow \lambda=1, \ -3$$

$\therefore$ The system has non – trivial solution only if $\lambda=1, \ -3$.

Case (i) : $\lambda=1 \Rightarrow A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} R_2=R_2-2R_1 \\ R_3=R_3+R_1 \end{pmatrix}$

$\Rightarrow x_1-2x_2+x_3=0$; Taking $x_2=c_1, \quad x_3=c_2$, we get , $x_1=2c_1-c_2$;

$\therefore \quad X=\left[(2c_1-c_2)c_1 \ c_2\right]^1=\left[x_1 \ x_2 \ x_3\right]^1$ is the general solution.

Case (ii): $\lambda=-3 \Rightarrow A=\begin{bmatrix} 5 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & 2 \\ 5 & -2 & 1 \end{bmatrix} (R_1 \leftrightarrow R_3)$

$$\sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 4 & 8 \\ 0 & 8 & 16 \end{bmatrix} \begin{pmatrix} R_2=R_2+2R_1 \\ R_3=R_3+5R_1 \end{pmatrix} \sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix} (R_3=R_3-2R_2)$$

$\Rightarrow -x_1+2x_2+3x_3=0; \quad x_2+2x_3=0;$ Taking $x_3=k$, we get $x_2=-2k$;
$x_1=2x_2+3x_3=-k$

$\therefore \ X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=k\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ where k is an arbitrary constant is the solution.

47. Determine the rank of the matrix $A=\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ by reducing it to normal form.

[JNTU 2005]

Solution

$$A=\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} (R_1 \leftrightarrow R_2)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{pmatrix} R_2=R_2+2R_1 \\ R_3=R_3-R_1 \end{pmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{pmatrix} c_2 = c_2 - 2c_1 \\ c_3 = c_3 - 3c_1 \\ c_4 = c_4 + c_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \left(R_2=R_2\times\frac{1}{3}\right) \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} R_3=R_3+2R_2 \\ R_4=R_4-R_2 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} c_3 = c_3 - c_2 \\ c_4 = c_4 - c_2 \end{pmatrix} \sim \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \text{ which is normal form.}$$

$\therefore$ Rank of A = 2

48. Determine the values of λ for which the following set of equations may possess nontrivial solution and solve them in each case.

$3x_1 + x_2 - \lambda x_3 = 0$; $4x_1 - 2x_2 - 3x_3 = 0$; $2\lambda x_1 + 4x_2 + \lambda x_3 = 0$.

[JNTU 2005]

Solution

The given system of equations can be written as,

$$\begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{........(1)} \quad \Rightarrow \text{AX} = 0\text{, where,}$$

$$\text{A}=\begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}; \text{ (1) Possesses non – trivial solution iff } |\text{ A }| = 0$$

$$\Rightarrow 3(-2\lambda+12)-1(4\lambda+6\lambda)-\lambda(16+4\lambda)=0 \Rightarrow 4\lambda^2+32\lambda-36=0$$

$$\Rightarrow \lambda^2+8\lambda-9=0 \quad \Rightarrow (\lambda+9)(\lambda-1)=0 \quad \Rightarrow \lambda=1,-9$$

$$\text{Case (i): } \lambda=1 \Rightarrow \begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} 3x_1+x_2-x_3=0 \\ 4x_1-2x_2-3x_3=0 \\ 2x_1+4x_2+x_3=0 \end{array} \quad \text{(from (1))}$$

Solving first two equations, we get, $\dfrac{x_1}{-5}=\dfrac{x_2}{5}=\dfrac{x_3}{-10} \Rightarrow \dfrac{x_1}{1}=\dfrac{x_2}{-1}=\dfrac{x_3}{2}=c_1$

where c_1 is arbitrary constant.

$$\therefore \text{ X}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} c_1 \\ -c_1 \\ 2c_1 \end{bmatrix} \quad \left(\text{Note the values of } x_1, x_2, x_3 \text{satisfy the 3rd equation}\right)$$

$$\text{Case (ii): } \lambda=-9 \Rightarrow \begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & -3 \\ -18 & 4 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} 3x_1+x_2+9x_3=0 \\ 4x_1-2x_2-3x_3=0 \\ -18x_1+4x_2-9x_3=0 \end{array} \quad \text{(from (1))}$$

The first two equations $\Rightarrow$

$$\frac{x_1}{15}=\frac{x_2}{45}=\frac{x_3}{-10}\Rightarrow\frac{x_1}{3}=\frac{x_2}{9}=\frac{x_3}{-2}\Rightarrow c_2\Rightarrow X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}3c_2\\9c_2\\-2c_2\end{bmatrix}$$

Note that these value satisfy 3rd equation also.

49. Find whether the following equations are consistent .If so solve them.

$2x-y-z=2$; $x+2y+z=2$; $4x-7y-5z=2$ [JNTU 2005]

Solution

The given system is $\begin{bmatrix}2 & -1 & -1\\1 & 2 & 1\\4 & -7 & -5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}\Rightarrow AX=B$

Augmented matrix $[A,B]=\begin{bmatrix}2 & -1 & -1 & 2\\1 & 2 & 1 & 2\\4 & -7 & -5 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 1 & 2\\2 & -1 & -1 & 2\\4 & -7 & -5 & 2\end{bmatrix}(R_1\leftrightarrow R_2)$

$$\sim\begin{bmatrix}1 & 2 & 1 & 2\\0 & -5 & -3 & -2\\0 & -15 & -9 & -6\end{bmatrix}\begin{pmatrix}R_2=R_2-2R_1\\R_3=R_3-4R_1\end{pmatrix}\sim\begin{bmatrix}1 & 2 & 1 & 2\\0 & -5 & -3 & -2\\0 & 0 & 0 & 0\end{bmatrix}(R_3=R_3-3R_2)=C,$$

which is Echelon form with two non – zero rows.

∴ Rank of A = Rank of [A ,B] = 2 < 3 (no of unknowns)

∴ The system is consistent and has infinite solutions involving (3 – 2) = 1 arbitrary constant.

From the matrix C, we get, $x+2y+z=2$; $+5y+3z=2$,

Taking $y=k$, we have $z=\frac{1}{3}(2-5k)$; $x=2-2k-\frac{1}{3}(2-5k)$

i.e., $x=\frac{4}{3}-\frac{k}{3}=\frac{1}{3}(4-k) \Rightarrow X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 4-k \\ 3k \\ 2-5k \end{bmatrix}$ 'k' being arbitrary constant.

50. Find whether the following set of equations are consistent .If so solve them.

$2x-y+3z-9=0$; $x+y+z=6$; $x-y+z-2=0$ [JNTU 2006]

Solution

If the system is written in the form $AX=B$, we have,

$$A=\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix};\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix};\ B=\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

Augmented matrix, $[A,B]=\begin{bmatrix} 2 & -1 & 3 & 9 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 9 \\ 1 & -1 & 1 & 2 \end{bmatrix} (R_1 \leftrightarrow R_2)$

$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & -3 \\ 0 & -2 & 0 & -4 \end{bmatrix} \begin{pmatrix} R_2=R_2-2R_1 \\ R_3=R_3-R_1 \end{pmatrix} = C;\ \because \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{vmatrix} = 2 \neq 0$, we infer that

Rank of A = Rank of [A,B] = 3 (number of unknowns)

$\therefore$ The system is consistent and has unique solution

$C \Rightarrow x+y+z=6$; $-3y+z=-3$; $-2y=-4$;

$\therefore$ $y=2$; $z=3$; $x=1$; is the solution.

Objective Type Questions

I. Choose the Correct Answer

1. If $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 4 & 5 \\ -2 & 6 & 9 \end{bmatrix}$, Trace of A =

   (1) 12 (2) 13 (3) 14 (4) 5

   [*Ans:* (3)]

2. The elements of the principal diagonal of a skew – symmetric are all

   (1) Equal to zero (2) Irrational (non zero) numbers

   (3) Equal to unity (4) Non–zero complex numbers.

   [*Ans:* (1)]

3. If $A^* = A$, A is called

   (1) a Symmetric matrix (2) a Skew symmetric matrix

   (3) a Hermitian matrix (4) a skew Hermitian matrix

   [*Ans:* (3)]

4. If $A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, AB =

   (1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 3 & 2 \\ -1 & -4 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (4) None

   [*Ans:* (2)]

5. 'A' is an orthogonal matrix. Then A^{-1} =

   (1) A^2 (2) A (3) A^* (4) A^T

   [*Ans:* (4)]

6. If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a periodic matrix, its period is equal to

   (1) 3 (2) 4 (3) 1 (4) none of these

   [*Ans:* (2)]

7. The rank of A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -1 & 0 & 0 \end{bmatrix}$ is

(1) 0 (2) 1 (3) 2 (4) 3

[*Ans:* (4)]

8. If A = $\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 3 & 5 & -2 \\ 3 & 4 & 6 & 8 \end{bmatrix}$, the rank of A can not exceed

(1) 2 (2) 3 (3) 4 (4) None

[*Ans:* (2)]

9. If A = $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & 5 \end{bmatrix}$ is subjected to the elementary transformation $R_{32(3)}$, the resulting matrix is

(1) $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 3 \\ 0 & 3 & 14 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(3) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & -2 & 1 \\ 3 & 1 & 5 \end{bmatrix}$ (4) none of these

[*Ans:* (1)]

10. If A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & a \\ -1 & 0 & 1 \end{bmatrix}$ and ρ (A) = 2, the value of 'a' is =

(1) 0 (2) 3 (3) 6 (4) −1

[*Ans:* (3)]

11. If $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$, $A^{-1} =$

(1) $\frac{1}{5}\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}$ (2) $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$

(3) $\frac{1}{5}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$ (4) $\frac{1}{5}\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

[*Ans:* (1)]

12. If the rank of $A = \begin{bmatrix} 1 & 3 & -1 \\ a & -2 & 0 \\ 3 & 2 & -1 \end{bmatrix}$ is 3, the value of 'a' is

(1) 3 (2) 5 (3) $\neq 4$ (4) 1

[*Ans:* (3)]

13. AX = B is a system of 'n' non homogeneous equations such that r(A/B) = r(A) = n, then, the system has

(1) no solution (2) a unique solution

(3) infinitely many solutions (4) None of these

[*Ans:* (2)]

14. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 14 \\ 2 \end{bmatrix}$, the solution matrix 'X' of the system of equation AX = B is

(1) $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ (2) $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ (3) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

[*Ans:* (4)]

15. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A^{-1} =$

(1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

(3) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (4) None of these

[*Ans:* (1)]

16. The solution of the system $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 9 & 1 \\ 8 & -27 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is

(1) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (2) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (3) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (4) None of these

[*Ans:* (3)]

17. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 3 & 1 \end{bmatrix}$, $A^{-1} =$

(1) $\frac{1}{10}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \\ 7 & -1 & 2 \end{bmatrix}$ (2) $\frac{1}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(3) $\frac{1}{9}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ (4) Does no exist

[*Ans:* (4)]

18. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and, $A^2 + 2A + I = 4B$, B =

(1) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

[*Ans:* (1)]

19. If $A = \begin{bmatrix} 1 & 2 & a \\ -1 & 1 & 3 \\ 2 & -1 & 1 \end{bmatrix}$ is a singular matrix, the value of 'a' is

(1) 9 (2) 18 (3) 6 (4) 3

[*Ans:* (2)]

20. The inverse of a singular matrix

(1) is orthogonal (2) is unit matrix

(3) is nonsingular (4) does not exist

[*Ans:* (4)]

II. Fill in the blanks

1. The order of $A = \begin{bmatrix} 2 & 4 & 5 & 6 \\ 1 & 0 & -1 & 3 \end{bmatrix}$ is ____________

[*Ans:* 2 by 4]

2. In a scalar matrix, the elements of principal diagonal are all ____________

[*Ans:* equal]

3. The sum of the elements of the principal diagonal of a square matrix is called its ____________

[*Ans:* trace]

4. The value of the determinant of an orthogonal matrix is equal to ____________

[*Ans:* ± 1]

5. If 'A' is a symmetric matrix, A^T = ____________

[*Ans:* 'A']

6. If $A^* = -A$ then 'A' is called ____________

[*Ans:* skew–Hermitian]

7. If 'A' is a square matrix such that $A^2 = A$ then 'A' is said to be an ____________ matrix.

[*Ans:* idempotent]

8. If 'A' is a square matrix such that $A^n = 0$ for a + ve integer 'n', 'A' is called a ______________ matrix and the least + ve integer 'n' satisfying $A^n = 0$ is called its ________________

[*Ans:* nilpotent ; index]

9. If 'A' is square matrix such that $A^{n+1} = A$ where 'n' is a + ve integer, 'A' is called a __________________ matrix, and the least + ve integer 'n' satisfying $A^{n+1} = A$ is called its __________________

[*Ans:* periodic ; period]

10. If $A^2 = I$ ('A' and 'I' are of same order), 'A' is called an ________________ matrix.

[*Ans:* involutary]

11. 'A' is a complex matrix and $A^* A = I$; then 'A' is said to be ________________

[*Ans:* unitary]

12. If 'A' is a square matrix such that $A^T = A^{-1}$, 'A' is said to be an ______________ matrix.

[*Ans:* orthogonal]

13. The product of 2 orthogonal matrices is ____________________

[*Ans:* orthogonal]

14. Express $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ as the sum of an upper triangular matrix and a lower triangular matrix with diagonal elements zero ____________________

[*Ans.* $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix}$]

15. If $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & -3 & 3 \end{pmatrix}$, rank of A = ________________

[*Ans:* 1]

16. The solution of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}$ is ________________

$\left[\textbf{\textit{Ans:}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$

17. If the system of 'n' homogeneous equations AX = 0 when 'A' = $(a_{ij})_{n \times n}$ has the trivial solution as the unique solution, then A is ________________.

[***Ans:*** nonsingular]

18. If A = $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 6 \\ 2 & 5 & x \end{bmatrix}$, and Trace of A = 10, then x = ________________

[***Ans:*** 5]

19. If A = $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 1 \\ -4 & 5 & 6 \end{bmatrix}$, cofactor of '6' = ________________

[***Ans:*** 3]

20. Under the elementary transformation $C_{23(-1)}$, the matrix A = $\begin{bmatrix} 1 & -4 & -1 \\ 2 & 3 & 2 \\ 3 & 1 & 5 \end{bmatrix}$ changes to ________________

$\left[\textbf{\textit{Ans:}} \begin{pmatrix} 1 & -3 & -1 \\ 2 & 1 & 2 \\ 3 & -4 & 5 \end{pmatrix} \right]$

III Indicate whether the following statements are 'True' or 'False'.

1. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, A^{-1} does not exist

[*Ans:* False]

2. $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 7 \\ 0 & 0 & 5 \end{bmatrix}$ is an upper triangular matrix. [*Ans:* True]

3. If 'A' is a square matrix, $(A, A^T)^T = A A^T$

[*Ans:* True]

4. If $A = \begin{bmatrix} 2+i & 1-i & 2 \\ 5-i & i & -1 \\ 4-2i & 1+i & -i \end{bmatrix}$, then $A^* = \begin{bmatrix} 2-i & 1+i & 2 \\ 5+i & -i & -1 \\ 4+2i & 1-i & i \end{bmatrix}$. [*Ans:* False]

5. If 'A' is a Hermitian matrix, $A^* = A$. [*Ans:* True]

6. $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ is skew symmetric. [*Ans:* True]

7. If 'A' is a skew– Hermitian matrix $A^* - A = 0$ [*Ans:* False]

8. If 'A' a square matrix such the $A^2 = I$ (I and A are of the same order), 'A' is said to be unitary.

[*Ans:* False]

9. $A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{bmatrix}$ is a periodic matrix of period '2' [*Ans:* False]

10. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\alpha & \cos\alpha \\ 0 & -\cos\alpha & \sin\alpha \end{bmatrix}$ is orthogonal

[*Ans:* True]

11. If 'A' is an orthogonal matrix, the dot product of any Two row or column vectors of 'A' is zero

[*Ans:* True]

12. A square matrix can be expressed as the sum of a symmetric and a skew–symmetric matrix in many ways.

[*Ans:* False]

13. If 'A' is any given matrix of rank 'r', the Rank of $A^T = r$

[*Ans:* True]

14. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, Rank of A = 0

[*Ans:* False]

15. $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & a & -2 \\ 1 & 0 & 0 \end{bmatrix}$, the rank of 'A' is 3 for all values of 'a'

[*Ans:* False]

16. $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is an example of canonical from

[*Ans:* True]

17. The system of equations $\begin{bmatrix} 2 & 1 & 5 \\ 3 & -2 & 2 \\ 5 & -8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is consistent.

[*Ans:* False]

18. The system of equations $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, has infinitely many solutions.

[*Ans:* True]

19. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & -4 & 5 \end{bmatrix}$, $A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 3 & 4 \\ -1 & 2 & 0 \\ -2 & 1 & -4 \end{bmatrix}$

[*Ans:* False]

20. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $A^2 - 3A + 2I = 0$

[*Ans:* False]