CHAPTER 1

PARTIAL FRACTIONS

1.1 INTRODUCTION

We know the method of combining number of fractions into a single by taking LCM (least common multiple) of denominators of the fractions. Now, here we have to learn, converse problem, that is splitting up of a given fraction into a number of simpler fractions called **Partial Fractions**.

In this Chapter, we study the method of resolving a given fraction into Partial fractions.

1.1.1 Polynomial

An algebraic expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ is called a polynomial in x, where $a_0 \neq 0$, a_1, a_2, … a_n are real constants and x is an unknown variable. The highest power of x that exists in the expression is called the degree of the polynomial.

Ex: $f(x) = x^5 + 4x^4 + 3x^3 + 4x^2 + x + 1$

$a_0 = 1$ $a_1 = 4$ $a_2 = 3$ $a_3 = 4$ $a_5 = 1$

$a_6 = 1$ are real constants and degree is 5, because the highest power of x is 5

1.1.2 Rational Fraction

The quotient $\dfrac{P(x)}{Q(x)}$ of two polynomials P(x) and $Q(x) \neq 0$ is called a rational fraction.

Ex: $P(x) = x^2 + 3x + 5$

$Q(x) = 3x^3 + 4x^2 + 5x + 1$

$$\frac{P(x)}{Q(x)} = \frac{x^2 + 3x + 5}{3x^3 + 4x^2 + 5x + 1}$$

1.1.3 Proper and Improper Fractions

The rational fraction $\frac{P(x)}{Q(x)}$ is a proper fraction, if the degree of the numerator P(x) is less than the degree of the denominator Q(x).

Ex: $\frac{x+1}{x^2 + 2x + 1} = \frac{P(x)}{Q(x)}$

$P(x) = x + 1$ – Degree = 1

$Q(x) = x^2 + 2x + 1$ – Degree = 2

The rational fraction $\frac{P(x)}{Q(x)}$ is a improper fraction, if the degree of numerator P(x) is greater than or equal to the degree of the denominator Q(x).

Ex: 1. $\frac{P(x)}{Q(x)} = \frac{x^2 + 2x + 1}{x^2 + 3x + 1}$

P(x) – degree = 2

Q(x) – degree = 2

2. $\frac{P(x)}{Q(x)} = \frac{x^3 + 2x^2 + 3x + 1}{x^2 + 3x + 2}$

P(x) – degree = 3

Q(x) – degree = 2

3. $\dfrac{x^3+2x^2+3x-4}{x^2+x+3}$

$$
\begin{array}{r|l}
 & \quad x+1 \\ \hline
x^2+x+3 & x^3+2x^2+3x-4 \\
 & x^3+x^2+3x \\ \hline
 & \quad x^2+0-4 \\
 & \quad x^2+x+3 \\ \hline
 & \quad -x-7
\end{array}
$$

$$\therefore \quad \frac{x^3+2x^2+3x-4}{x^2+x+3} = x+1+\frac{-x-7}{x^2+x+3}$$

1.2 PARTIAL FRACTION

Consider

$$\frac{1}{x+1}+\frac{2}{x+2}=\frac{x+2+2(x+1)}{(x+1)(x+2)}$$

$$\frac{1}{x+1}+\frac{2}{x+2}=\frac{3x+3}{(x+1)(x+2)}$$

How to resolve $\dfrac{3x+3}{(x+1)(x+2)}$ into $\dfrac{1}{x+1}+\dfrac{2}{x+2}$

1.2.1 Rules to Resolve into Partial Fraction

1. Denominator containing linear distinct factors

$$\frac{p(x)}{q(x)r(x)}=\frac{A}{q(x)}+\frac{B}{r(x)}$$

A and B are constants to be determined

$$\frac{3x+3}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2} \qquad \ldots\ldots (1)$$

Take LCM for RHS – side equation

$$\frac{3x+3}{(x+1)(x+2)}=\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

Cancel the denominator of LHS and RHS

$3x + 1 = A(x + 2) + B(x + 1)$ ….. (2)

Put $x + 1 = 0 \rightarrow x = -1$

Substitute $x = -1$ in equation (2)

$3(-1) + 1 = A(-1 + 2) + B.0$

$-3 + 1 = A.1 + 0$

$-2 = A, A = -2$

$A = -2$

Put $x + 2 = 0 \rightarrow x = -2$

Substitute $x = -2$ in equation (2)

$3x + 1 = A(x + 2) + B(x + 1)$

$(-2) + 1 = A.0 + B(-2 + 1)$

$-6 + 1 = 0 + B(-1)$

$-5 = -B \rightarrow B = 5$

$B = 5$

Take equation (1), substitute A and B

$$\frac{3x+3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\frac{3x+3}{(x+1)(x+2)} = \frac{-2}{x+1} + \frac{5}{x+2}$$

2. Denominator containing linear repeated factors:

If $\frac{p(x)}{q(x)} = \frac{x+1}{(x+a)^3}$

The fraction should be resolve as

$$\frac{x+1}{(x+a)^3} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3}$$

Then the values of A, B, C are to be determined

Ex: $$\frac{x+1}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \qquad \text{.....(1)}$$

Take terms LCM and add the RHS side

$$\frac{x+1}{(x+2)^3} = \frac{A(x+2)^2 + B(x+2) + C}{(x+2)^3}$$

$$x + 1 = A(x + 2)^2 + B(x + 2) + C \qquad \text{.....(2)}$$

Put $x + 2 = 0$

$x = –2$

Substitute $x = –2$ in equation (2)

$–2 + 1 = 0 + 0 + C$

$–1 = C$

$C = –1$

$x + 1 = A(x + 2)^2 + B (x + 2) + C$

$x = 1 = A(x^2 + 4x + 4) + B(x + 2) + C$

$x + 1 = Ax^2 + 4Ax + 4A + Bx + 2B + C$

$x + 1 = Ax^2 + 4Ax + 4A + Bx + 2B + C$

Equate coefficient of x

$$1 = 4A + B \qquad \text{.....(3)}$$

Equate coefficient of x^2

$0 = A$

$A = 0$

Take (3)

$1 = 4.A + B$

$1 = 4.0 + B$

$B = 1$

Substitute A, B and C in (1)

$$\frac{x+1}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

$$= \frac{0}{x+2} + \frac{1}{(x+2)^2} + \frac{-1}{(x+2)^3}$$

$$\frac{x+1}{(x+2)^3} = \frac{1}{(x+2)^2} - \frac{1}{(x+2)^3}$$

3. Denominator containing non-repeated non-factorizable factor. To each non-repeated, non-factorizable quadratic factor of the type $(ax^2 + bx + c)$ of the denominator,

Ex: $$\frac{x^2-1}{(x^2+3)(x+1)} = \frac{Ax+B}{x^2+3} + \frac{C}{x+1} \quad \text{..... (1)}$$

Take L.C.M

$$\frac{x^2-1}{(x^2+3)(x+1)} = \frac{(Ax+B)(x+1)+C(x^2+3)}{(x^2+3)(x+1)}$$

$$x^2 - 1 = (Ax + B)(x + 1) + C\,(x^2 + 3) \quad \text{.....(2)}$$

Put $x + 1 = 0 \rightarrow x = -1$

Substitute $x = -1$ in (2)

$x^2 - 1 = (Ax + B)\,(x + 1) + C\,(x^2 + 3)$

$(-1)^2 - 1 = 0 + ((-1)^2 + 3)\,C$

$0 = (1 + 3)\,C$

$0 = 4.C \qquad C = 0$

To find $A \propto B$

Equate coefficient x in the equation

$x^2 - 1 = (Ax + B)(x + 1) + C\,(x^2 + 3)$

$0 = A + B$

$A + B = 0 \qquad A = -B$

Compare coefficient of x^2

$1 = A + C$

$1 = A + 0$

$A = 1$

But

$A = -B$

$B = -A$

$\therefore \quad B = -1$

$$\frac{x^2-1}{(x^2+3)(x+1)} = \frac{Ax+B}{x^2+3} + \frac{C}{x+1}$$

Substitute A = 1, b = –1, c = 0 in equation (1)

$$\frac{x^2-1}{(x^2+3)(x+1)} = \frac{1.x-1}{x^2+3} + \frac{0}{x+1}$$

$$\frac{x^2-1}{(x^2+3x+1)} = \frac{x-1}{x^2+3}$$

$$= \frac{x}{x^2+3} - \frac{1}{x^2+3}$$

1.3 APPLICATION OF PARTIAL FRACTION IN CHEMICAL KINETICS AND PHARMACOKINETICS

1. In a second-order kinetics, the differential rate expression is given by the equation

$$r = \frac{dx}{dt} = K_2(a-x)(b-x) \qquad \frac{dx}{(a-x)(b-x)} = Kdt$$

To solve this equation. We have to obtain the partial fraction of

$$\frac{1}{(a-x)(b-x)}$$

As per Partial fraction rule

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x} \qquad \text{.....(1)}$$

Take LCM

$$\frac{1}{(a-x)(b-x)} = \frac{A(b-x)+B(a-x)}{(a-x)(b-x)}$$

$$1 = A(b-x) + B(a-x) \qquad \text{.....(2)}$$

Put $b - x = 0 \quad x = b$

$1 = 0 + B\,(a - b)$

$$\frac{1}{a-b} = B$$

Put $a - x = 0, x = a$

Take equation (2)

$$1 = A\,(b-x) + B\,(a-x)$$

$$1 = A\,(b-a) + .0$$

$$A = \frac{1}{b-a}$$

But

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$$

Substitute A and B in equation (1)

$$\frac{1}{(a-x)(b-x)} = \frac{\frac{1}{b-a}}{a-x} + \frac{\frac{1}{a-b}}{b-x}$$

$$= \frac{1}{(b-a)(a-x)} + \frac{1}{(a-b)(b-x)}$$

$$= \frac{1}{-(a-b)(a-x)} + \frac{1}{(a-b)(b-x)}$$

$$= \frac{1}{(a-b)(b-x)} - \frac{1}{(a-b)(a-x)}$$

$$= \frac{1}{a-b}\left[\frac{1}{b-x} - \frac{1}{a-x}\right]$$

2. First order Absorption (Infusion Method)

For a drug that enters the body by an apparent first-order absorption process, is eliminated by a first - order process and distributes in the body according to a one - compartment model, is given by equation.

$$\frac{dx}{dt} = K_a X_a - KX$$

It can be reduced into the form $\overline{X} = \frac{K_a F X_0}{(S+K)(S+K_a)}$ by applying Laplace transform

To find the value of X, we have to find the partial fraction of the equation

$$\frac{1}{(S+K)(S+K_a)}$$

$$\frac{1}{(S+K)(S+K_a)} = \frac{A}{S+K} + \frac{B}{S+K_a}$$

Take LCM in RHS

$$\frac{1}{(S+K)(S+K_a)} = \frac{A(S+K_a)+B(S+K)}{(S+K)(S+K_a)}$$

$$1 = A(S + K_a) + B(S + K) \qquad \text{.....(1)}$$

Put S + K = 0 $\rightarrow$ S = –K

Substitute S = –K in (1)

$$\frac{1}{K_a - K} = A$$

Put $S + K_a = 0 \rightarrow S = -K_a$

Substitute $S = -K_a$ in (1)

$$1 = A (S + K_a) + B (S + K)$$

$$1 = A - 0 + B (-K_a + K)$$

$$B = \frac{1}{K - K_a}$$

$$B = \frac{1}{(K - K_a)}$$

Take

$$\frac{1}{(S+K)(S+K_a)} = \frac{A}{S+K} + \frac{B}{S+K_a}$$

Substitute the values A and B

$$\frac{1}{(S+K)(S+K_a)} = \frac{\frac{1}{K_a - K}}{S+K} + \frac{\frac{1}{K-K_a}}{S+K_a}$$

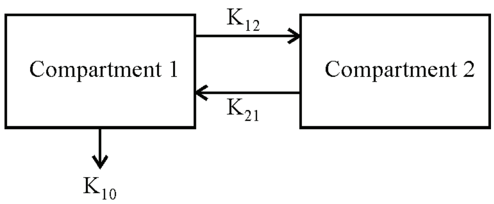
$$= \frac{1}{(K_a - K)(S+K)} + \frac{1}{(K-K_a)(S+K_a)}$$

$$= \frac{1}{(K_a - K)(S+K)} - \frac{1}{(K_a - K)(S+K_a)}$$

$$\frac{1}{(S+K)(S+K_a)} = \frac{1}{K_a - K}\left[\frac{1}{S+K} - \frac{1}{S+K_a}\right]$$

3. The rapid intravenous injection of a drug that distribute in the body according to two compartment model system with elimination occurring from the central compartment is as shown in model and the amount of drug in the central compartment a_{sc} is given by equation (function)

$$a_{sc} = X_0 \frac{(S+E_2)}{(S+\lambda_1)(S+\lambda_2)}$$



To calculate concentration of drug in the central compartment, the above mentioned equation $\frac{(S+E_2)}{(S+\lambda_1)(S+\lambda_2)}$ has to be resolved into partial fraction.

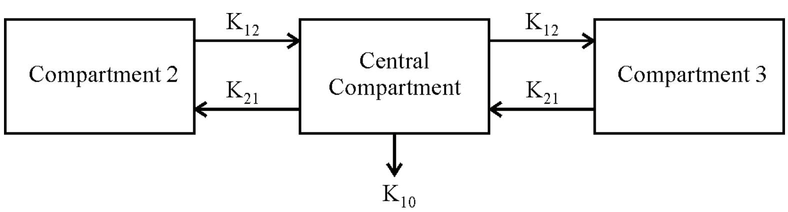
After resolution into partial fraction, we will arrive into the equation

$$\frac{(S+E_2)}{(S+\lambda_1)(S+\lambda_2)} = \frac{A}{S+\lambda_1} + \frac{B}{S+\lambda_2}$$

$$= \frac{E_2-\lambda_1}{(\lambda_2-\lambda_1)(S+\lambda_1)} - \frac{E_2-\lambda_2}{(\lambda_2-\lambda_1)(S+\lambda_2)}$$

4. The disposition function for the central compartment in three compartmental model is given by the equation.

$$d_{sc} = \frac{(S+E_2)(S+E_3)}{(S+\lambda_1)(S+\lambda_2)(S+\lambda_3)}$$



Amount of drug in central compartment $a_{sc,}$ which is product of input and disposition is given by the fraction.

$$a_{sc} = \frac{X_0(S+E_2)(S+E_3)}{(S+\lambda_1)(S+\lambda_2)(S+\lambda_3)}$$

After resolving into partial fraction, the equation will be reduced into the form

$$a_{sc} = \frac{X_0(E_2-\lambda_1)(E_3-\lambda_1)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\frac{1}{(S+\lambda_1)} + \frac{X_0(E_2-\lambda_2)(E_3-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\frac{}{(S+\lambda_2)}$$

$$+\frac{X_0(E_2-\lambda_3)(E_3-\lambda_3)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(S+\lambda_3)}$$

5. When the drug is administered intravenously at a constant rate, rate of change of drug in the body with respect time can be obtained by the Linear differential equation.

$$\frac{dx}{dt} = K_0 - Kx$$

K_0 – is rate of drug infusion

After simplification by applying Laplace Transform

$$\overline{X} = \frac{K_0}{S(S+K)}$$

To calculate total amount of drug in the body, the equation $\frac{K_0}{S(S+K)}$ has to be resolved into partial fraction form

$$\frac{K_0}{S(S+K)} = \frac{A}{S} + \frac{B}{S+K}$$

Take LCM

$$\frac{1}{S(S+K)} = \frac{A(S+K)+B(S)}{S(S+K)}$$

$$1 = A(S + K) + B.S \qquad \text{..... (1)}$$

Put $S = 0$ in (1)

$$1 = A(0 + K) + B.0$$

$$1 = A.K$$

$$A = \frac{1}{k}$$

Put $S + K = 0 \rightarrow S = -K$

Substitute $S = -K$ in equation (1)

$$1 = A\,(S + K) + B.S$$

$$1 = A.0 + B(-K)$$

$$B = -\frac{1}{K}$$

$$\frac{K_0}{S(S+K)} = \frac{A}{S} + \frac{B}{S+K}$$

$$\frac{1}{S(S+K)} = \frac{\frac{1}{K}}{S} + \frac{\frac{-1}{K}}{S+K}$$

$$= \frac{1}{K}.\frac{1}{S} - \frac{1}{K}\frac{1}{S+K}$$

Exercise 1.1

Resolve into Partial Fraction

1. $\dfrac{3x+5}{(x-1)(x+2)}$
2. $\dfrac{1}{(x-2)(x-1)^2}$
3. $\dfrac{2x-3}{(x+5)(x-2)}$
4. $\dfrac{x+1}{(x)(x+2)(x+4)}$
5. $\dfrac{3x^2+2x-2}{(x-1)^2(2x-1)}$
6. $\dfrac{5x^2+1}{x^3-1}$

Answers

1. $\dfrac{8}{3(x-1)} + \dfrac{1}{3(x+2)}$
2. $\dfrac{1}{(x-2)} - \dfrac{1}{(x-1)} - \dfrac{1}{(x-1)^2}$
3. $\dfrac{13}{7(x+5)} + \dfrac{1}{7(x-2)}$
4. $\dfrac{-1}{2x} - \dfrac{2}{3(x-1)} - \dfrac{1}{6(x+2)}$
5. $\dfrac{2}{x-1} + \dfrac{3}{(x-1)^2} - \dfrac{1}{2x-1}$
6. $\dfrac{2}{(x-1)} + \dfrac{3x+1}{(x^2+x+1)}$