Part I

CHAPTER 1

Matrices

1.1 Introduction

Matrix algebra is a powerful mathematical formulation in the context of solving linear algebraic equations, linear transformations and the solutions of various types of differential equations. It has also become a powerful tool to tackle the problems of rigid body motions, oscillations, transformation of different coordinate systems, development and mathematical formulations of Quantum mechanics and the theory of representation of groups.

Let us consider the following system of equations

$$x + 2y + 3z = 11$$

$$2x - y - z = -3$$

$$3x + 4y + 2z = 17$$

If we arrange the coefficients of x, y and z in the order in which they occur in the given equations and enclose them in brackets, we get the following rectangular array of numbers

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 3 & 4 & 2 \end{pmatrix} \text{ or } \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 3 & 4 & 2 \end{bmatrix} \text{ or } \begin{Vmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 3 & 4 & 2 \end{Vmatrix}$$

This type of rectangular array of numbers has been given the name matrix. The horizontal lines are called rows and the vertical lines are called columns.

1.2 Definition of Matrix

Matrix is a rectangular array of real or complex numbers in rows and columns. A matrix is denoted by the capital letters A, B, C etc. If there are m rows and n columns in the matrix, then the matrix is called a $m \times n$ matrix.

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & -4 \end{pmatrix}$ or $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & -4 \end{bmatrix}$

Here A is a 2 × 3 matrix because A has two rows and three columns.

Note:

(i) Although matrix is just a rectangular array of numbers but these numbers are enclosed within big or small brackets to make it good looking.

(ii) Matrix is just an array of numbers and has no numerical value as in case of a determinant.

(iii) Matrices are not necessarily square as determinants must be. A matrix is simply a display or table of values.

Elements of a matrix: The numbers occurring in the rectangular array (matrix) are called the elements of the matrix. The elements of the matrix denoted by the capital letters are usually denoted by the corresponding small letters with lower suffixes. Thus the element of the i^{th} row and j^{th} column of the matrix denoted by the capital letters A is usually denoted by the corresponding small letter a_{ij}. The matrix A is sometimes also written as (a_{ij}) or $[a_{ij}]$.

Types of Matrices

1. ***Square matrix:*** A matrix having equal number of rows and columns is called a square matrix. If the matrix A has n rows and n columns it is said to be a square matrix of order n.

   Example: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$ is a square matrix of order 3.

2. ***Horizontal matrix:*** A m × n matrix is called a horizontal matrix if m < n.

   Example: $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{pmatrix}$. Here A is a horizontal matrix.

3. ***Vertical matrix:*** A m × n matrix is called a vertical matrix if m > n.

   Example: $B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$. Here B is a vertical matrix.

4. ***Row matrix and Row vector:*** A matrix having only one row is called a row matrix or a row vector.

   Example: $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. Here A is a row matrix.

5. ***Column matrix or Column vector:*** A matrix having only one column is called a column matrix or column vector.

Example: $B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}$. Here B is a column matrix

6. ***Zero matrix or Null matrix:*** A matrix whose each element is a zero is called a zero matrix or null matrix.

Example: (i) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (ii) $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Here A and B are zero matrices.

Diagonal of matrix

Let A be a square matrix of order n.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Here the elements a_{11}, a_{22}, a_{33}, ..., a_{nn} are called the diagonal elements and the line along which these elements lie is called the principal diagonal of the matrix.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$

Here 1, 0, 4 are the diagonal elements of the matrix A.

7. ***Diagonal matrix:*** A square matrix having all elements not occurring along the principal diagonal equal to zero is called a diagonal matrix. Thus the square matrix (a_{ij}) is called the diagonal matrix if $a_{ij} = 0$ for $i \neq j$

Examples: (i) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Here A and B are diagonal matrices

8. ***Scalar matrix:*** A square matrix is said to be a scalar matrix if all elements along the principal diagonal are equal and all elements not occurring along the principal diagonal are zero.

Examples: (i) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Here A and B are scalar matrices.

9. ***Unit matrix:*** A square matrix is said to be a unit matrix if all the elements along the principal diagonal are unity (1) and all elements not occurring along the principal diagonal are zero.

Examples: (i) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (ii) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A unit matrix of order n is denoted by I_n.

Thus in the examples given above $A = I_2$, $B = I_3$.

10. ***Triangular matrix***

(a) *Lower triangular matrix*: A square matrix (a_{ij}) is called a lower triangular matrix if $a_{ij} = 0$ when $i < j$.

(b) *Upper triangular matrix*: A square matrix (a_{ij}) is called an upper triangular matrix if $a_{ij} = 0$ when $i > j$.

Examples: (i) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

(ii) $C = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Here A and B are lower triangular matrices.

Here C and D are upper triangular matrices

Submatrix of a matrix: Any matrix obtained by omitting some rows and some columns from the matrix A is called a submatrix of the matrix A.

Example: Let $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 0 & 3 & -1 \\ 4 & 3 & 2 & 0 \end{bmatrix}, B = \begin{pmatrix} 1 & 3 & 4 \\ 4 & 3 & 2 \end{pmatrix}$

Here B has been obtained by omitting 4th column and 2nd row from A. Hence B is a submatrix of A.

Note: A itself is a submatrix of A.

Equality of two matrices: Two m × n matrices A and B are said to be equal if the corresponding elements of the two matrices are equal.

i.e., if $a_{ij} = b_{ij}$ for i = 1, 2, …, m; j = 1, 2, 3, …, n

i.e., if $a_{11} = b_{11}$, $a_{12} = b_{12}$ etc.

Example: Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \end{pmatrix}$

Here A and B are equal matrices and we write A = B.

Addition of matrices: Let A and B be two m × n matrices. The m × n matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by A + B.

Thus if $A = (a_{ij}), B = (b_{ij}), \begin{matrix} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{matrix}$

then $A + B = (a_{ij}) + b_{ij}), \begin{matrix} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{matrix}$

Example: Let $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 4 & 5 \\ 6 & 2 & 9 \end{pmatrix}$

Then $A + B = \begin{pmatrix} 2-3 & 3+4 & 4+5 \\ 0+6 & 2+2 & -1+9 \end{pmatrix} = \begin{pmatrix} -1 & 7 & 9 \\ 6 & 4 & 8 \end{pmatrix}$

Note: A + B is defined only when they are matrices if the same type i.e., when the number of rows in A is equal to the number of rows in B and the number of columns in A is equal to the number of columns in B.

Properties of Matrix Addition

Property I: Matrix addition is commutative i.e., if A and B be any two m × n matrices, then A + B = B + A.

Proof: Let $A = (a_{ij})$ and $B = (b_{ij}); \begin{matrix} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{matrix}$

Then $A + B = (a_{ij} + b_{ij}); \begin{matrix} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{matrix}$

$$= (b_{ij} + a_{ij}); \quad \begin{matrix} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{matrix}$$

$$= (b_{ij}) + (a_{ij}); \quad \begin{matrix} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{matrix}$$

$$= B + A$$

Property II: Matrix addition is associative i.e., if A, B and C be three m × n matrices, then

$$A + (B + C) = (A + B) + C$$

Proof: Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be three m × n matrices.

Now $$A + (B + C) = (a_{ij}) + \{(b_{ij}) + (c_{ij})\}; \quad \begin{matrix} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{matrix}$$

$$= (a_{ij}) + (b_{ij} + c_{ij})$$

$$= (a_{ij} + \{b_{ij} + c_{ij}\})$$

$$= (\{a_{ij} + b_{ij}\} + c_{ij})$$

$$= (a_{ij} + b_{ij}) + (c_{ij})$$

$$= \{(a_{ij}) + (b_{ij})\} + (c_{ij})$$

$$= (A + B) + C$$

Property III: Cancellation laws hold good for addition of matrices i.e., if A, B, C, be any three m × n matrices, then

(i) $A + B = A + C \Rightarrow B = C$ (left cancellation law)

(ii) $B + A = C + A \Rightarrow B = C$ (right cancellation law)

Negative of a matrix: Let $A = (a_{ij})$ be m × n matrix. Then the negative of the matrix A is denoted by – A and is defined as $(-a_{ij})$.

In order to find – A sign of each element of A must be changed.

Example: Let $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -5 \end{pmatrix}$. Then $-A = \begin{pmatrix} -1 & 2 & 0 \\ -3 & -2 & 5 \end{pmatrix}$

Note: If A is a m × n matrix, then –A is also a m × n matrix

Subtraction of two matrices: Let A and B be two m × n matrices.

Then the difference of A and B is denoted by A – B and is defined by

$$A - B = A + (-B)$$

A – B will be also a m × n matrix. In order to find A – B, the elements of B must be subtracted from the corresponding elements of A.

Examples: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 & 4 \\ 5 & 0 & 6 \end{pmatrix}$

then $$A - B = \begin{pmatrix} 1-2 & 2+1 & 3-4 \\ 3-5 & -1-0 & 0-6 \end{pmatrix} = \begin{pmatrix} -1 & 3 & -1 \\ -2 & -1 & -6 \end{pmatrix}$$

Multiplication of a matrix by a scalar: Let A be any m × n matrix and k be any scalar (real or complex number), then the scalar multiple of matrix A by k is denoted by kA or Ak and is defined as the m × n matrix obtained by multiplying each element of A by k.

Thus if $A = (a_{ij});\ \begin{matrix} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{matrix}$

then $kA = (ka_{ij});\ \begin{matrix} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{matrix}$

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 5 & 4 \end{bmatrix}$, then $3A = \begin{bmatrix} 3 & 6 \\ 9 & 0 \\ 15 & 12 \end{bmatrix}$

Properties of scalar multiplication

(i) If A and B are any two m × n matrices and k is any scalar, then

$$k(A + B) = kA + kB$$

(ii) If A is any m × n matrix and a and b are any two scalars, then

$$(a + b) A = aA + bA$$

(iii) If A be any m × n matrix and k be any scalar, then

$$(-k)A = -(kA) = k(-A)$$

Multiplication of two matrices

Let $A = [a_{ij}]$ be a m × n matrix and $B = [b_{jk}]$ be a n × p matrix such that the number of columns in A is equal to the number of rows in B, then the m × p matrix $C = [c_{ik}]$ such that

$$C_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

$$= a_{i1} b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \ldots + a_{in}b_{nk}$$

is said to the product of the matrices A and B in that order and is denoted by AB.

Note:

(i) Product AB is defined only when the number of columns in A is equal to the number of rows in B.

(ii) If A is a m × n matrix and B is a n × p matrix, then AB will be a m × p matrix.

$$A : m \times \boxed{n}$$
$$B : \boxed{n} \times p$$
$$\overline{AB : m \times p}$$

(iii) Element of i^{th} row and k^{th} column of the product matrix is

$$a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \ldots + a_{in}b_{nk}$$

i^{th} row of A is $a_{i1}\ a_{i2}\ a_{i3} \ldots b_{in}$

k^{th} column of B is $b_{1k} b_{2k} b_{3k} \ldots b_{nk}$

Element of i^{th} row and k^{th} column of AB

$$= a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots + a_{in}b_{nk}$$

Thus in order to write down the element of the i^{th} row and k^{th} column of AB, we take the elements of i^{th} row of the first matrix A and multiply them with the corresponding elements of the k^{th} column of the second matrix B and then add them.

(iv) In the product AB, A is called the pre-factor and B is called the post-factor.

(v) If the product AB is possible then it is not necessary that the product BA is also possible.

(vi) If A be a m × n matrix and both AB and BA are defined then B will be a n × m matrix.

Examples:

(i) $$A = 3\times 2 \text{ matrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},\quad B = 2\times 2 \text{ matrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then $$AB = 3\times 2 \text{ matrix} \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

(ii) $$A = 3\times 3 \text{ matrix} \begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 4 \\ 3 & -2 & 6 \end{bmatrix},\quad B = 3\times 2 \text{ matrix} \begin{bmatrix} 4 & -1 \\ 2 & -2 \\ 5 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1.4 + 0.2 + 5.5 & 1(-1) + 0(-2) + 5.3 \\ (-1)4 + 2.2 + 4.5 & (-1)(-1) + 2(-2) + 4.3 \\ 3.4 + (-2)2 + 6.5 & 3(-1) + (-2)(-2) + 6.3 \end{bmatrix} = \begin{bmatrix} 29 & 14 \\ 20 & 9 \\ 38 & 19 \end{bmatrix}$$

Here BA is not defined because number of column in B is not equal to the number of rows in A.

Properties of matrix multiplication

(i) ***Matrix multiplication is associative:***

i.e., If A, B and C be m × m, n × p and p × q matrices, then

$$A(BC) = (AB)C$$

Proof: Let $A = [a_{ij}]_{m \times n}$, $B = [b_{jk}]_{n \times p}$ and $C = [c_{kl}]_{p \times q}$

B is a n × p matrix and C is a p × q matrix, therefore, BC will be a n × q matrix and since A is a m × n matrix, therefore, A (BC) will be a m × q matrix.

Again A is a m × n matrix and B is n × p matrix, therefore, AB will be m × p matrix and since C is a p × q matrix, therefore (AB) C is a m × q matrix.

Now element of i^{th} row and k^{th} column of AB

$$u_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

$$= a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk} \quad \ldots..(1.1)$$

Element of i^{th} row and l^{th} column of (AB) C

$$= \sum_{k=1}^{p} u_{ik} c_{kl} = \sum_{k=1}^{p} \left\{ \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) c_{kl} \right\} \quad \ldots..(1.2)$$

Again element of j^{th} row and l^{th} column of BC

$$v_{jl} = \sum_{k=1}^{p} b_{jk} c_{kl} \quad \ldots..(1.3)$$

Element of i^{th} row and l^{th} column of A (BC)

$$= \sum_{j=1}^{n} a_{ij} v_{jl} = \sum_{j=1}^{n} \left\{ a_{ij} \sum_{k=1}^{p} b_{jk} c_{kl} \right\}$$

$$= \sum_{k=1}^{p} \left\{ \left(\sum_{j=1}^{n} a_{ij}.b_{jk} \right) c_{kl} \right\}$$

From eq. 1.2 and 1.4 it follows that the element of i^{th} row and l^{th} column of A(BC) = element of i^{th} row and i^{th} column of (AB) C for all admissible values of i and *l*.

Hence $\quad A(BC) = (AB)C$

(ii) Multiplication of matrices is distributive with respect to addition of matrices i.e., if A is a m × n matrix and B and C are both n × p matrices.

then $\quad A (B + C) = AB + AC$

Proof: Let $A = [a_{ij}]_{m \times n}$, $B = [b_{jk}]_{n \times p}$ and $C = [c_{jk}]_{n \times p}$

Elements of i^{th} row of A are

$$a_{i1}, a_{i2}, \ldots, a_{in} \quad \ldots..(1.4)$$

Elements of k^{th} column of B are

$$b_{1k}, b_{2k}, \ldots, b_{nk} \quad \ldots..(1.5)$$

Elements of k^{th} column of C are

$$c_{1k}, c_{2k}, \ldots, c_{nk} \quad \ldots..(1.6)$$

$\therefore$ Elements of k^{th} column of (B + C) are

$$b_{1k} + c_{1k}, b_{2k} + c_{2k}, \ldots, b_{nk} + c_{nk} \quad \ldots..(1.7)$$

Elements of i^{th} row of A are

$$a_{i1}, a_{i2}, \ldots, a_{in} \quad \ldots..(1.8)$$

Element of i^{th} row and k^{th} column of AB

$$= a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk} \qquad \text{[From (1.4) and (1.5)]} \quad \ldots..(1.9)$$

Element of i^{th} row and k^{th} column of AC

$$= a_{i1}c_{1k} + a_{i2}c_{2k} + \ldots + a_{in}c_{nk} \qquad \text{[From (1.4) and (1.6)]} \quad \ldots..(1.10)$$

$\therefore$ Element of i^{th} row and k^{th} column of AB + AC

$$= (a_{il}b_{1k} + a_{il}c_{1k}) + (a_{i2}b_{2k} + a_{i2}c_{2k}) + \ldots + (a_{in}b_{nk} + a_{in}c_{nk}) \quad \ldots..(1.11)$$

Element of i^{th} row and k^{th} column of A (B + C)

$$= a_{il}(b_{1k} + a_{il}c_{1k}) + a_{i2}(b_{2k} + c_{2k}) + \ldots + a_{in}(b_{nk} + a_{in}c_{nk})$$

$$\text{[From (1.4) to (1.7)]} \quad \ldots..(1.12)$$

$$= (a_{il}b_{1k} + a_{il}c_{1k}) + (a_{i2}b_{2k} + a_{i2}c_{2k}) + \ldots + (a_{in}b_{nk} + a_{in}c_{nk}) \quad \ldots..(1.13)$$

From (1.12) and (11.3), we have

$$A(B + C) = AB + AC$$

Note: It can be also proved that

(i) if A and B are $m \times n$ matrices and C is a $n \times p$ matrix, then

$$(A + B)C = AC + BC$$

(ii) if A be a $m \times n$ matrix and I_n be the unit matrix of order n, then

$$AI_n = I_nA = A$$

Transpose of a matrix: Let A be any matrix then the matrix obtained by interchanging its rows and columns is called the transpose of A and is denoted by A' or A^T. If A is a m × n matrix then A' will be a n × m matrix.

Example: $A=\begin{pmatrix}1 & 2 & 3\\ 2 & 0 & 5\end{pmatrix}$, then $A'=\begin{bmatrix}1 & 2\\ 2 & 0\\ 3 & 5\end{bmatrix}$

Note: If $A=[a_{ij}]; \quad i=1,2,...,m, j=1,2,...,n$

then $A'=[a_{ji}]; \quad j=1,2,...,n,\ i=1,2,..,m$

Properties of Transpose of Matrices

Property I: $(A+B)'=A'+B'$

Proof: Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$

$\therefore \quad A+B=[a_{ij}+b_{ij}]$, then $(A+B)'=[a_{ji}+b_{ji}]_{n\times m}$ …..(1.14)

$A'=[a_{ij}]_{n\times m}$ and $B'=[b_{ji}]_{n\times m}$

$\therefore \quad A'+B'=[a_{ij}+b_{ji}]_{n\times m}$ …..(1.15)

From (1.14) and (1.15) it follows that $(A+B)=A'+B'$

Property II: If A is any matrix, then $(A')'=A$

Proof: Let $A=[a_{ij}]_{m\times n}$, then

$A'=[a_{ij}]_{n\times m} \quad \therefore\ (A')'=[a_{ij}]_{m\times n}=A$

Property III: If k is any number real or complex and A be any matrix, then

$(kA)'=kA'$

Proof: Let $A=[a_{ij}]_{m\times n}$, then $A'=[a_{ji}]_{n\times m}$

Now $kA=[ka_{ij}]_{m\times n}\ \therefore\ (kA)'=[ka_{ji}]_{n\times m}$ …..(1.16)

Also $kA'=[ka_{ji}]_{n\times m}$ …..(1.17)

From (1.16) and (1.17) it follows that $(kA)'=kA'$

Property IV: If A be a m × n matrix and B be a n × p matrix, then

$$(AB)' = B'A'$$

Proof: Let $A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times p}$

Then $A' = [a_{ji}]_{n\times m}$, $B' = [b_{kj}]_{n\times p}$

AB will be a m × p matrix, $B'A'$ will be a p × m matrix.

Element of the k^{th} row and i^{th} column of $(AB)'$

= element of i^{th} row and k^{th} column of AB

$$= \sum_{j=1}^{n} a_{ij} b_{jk} \qquad \ldots..(1.18)$$

Element of k^{th} row and i^{th} column of $B'A'$

$$= \sum_{j=1}^{n} b_{jk} a_{ij} = \sum_{j=1}^{n} a_{ij} b_{jk} \qquad \ldots..(1.19)$$

$\therefore$ [elements of k^{th} row of B' = elements of k^{th} column of B]

They are : $b_{1k}, b_{2k}, \ldots, b_{nk}$

and elements of i^{th} column of A' = element of i^{th} row of A

They are: $a_{i1}, a_{i2}, \ldots, a_{in}$

From (1.18) and (1.19) it follows that $(AB) = B'A'$

Symmetric, Skew Symmetric and Orthogonal Matrices

(i) ***Symmetric matrix:*** A square matrix A = $[a_{ij}]$ is said to be a symmetric matrix if

$$a_{ij} = a_{ji} \text{ for all i and j}$$

Example: $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 0 \\ 5 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix} \because$

***Note*:** A square matrix A is symmetric if and only if $A' = A$.

(ii) ***Skew symmetric matrix:*** A square matrix A = $[a_{ij}]$ is said to be a skew symmetric matrix if $a_{ij} = -a_{ji}$ for all i and j.

$\because$ $a_{ij} = -a_{ji}$ for all i and j

$\therefore$ $a_{ii} = -a_{ii}$ [putting j = i]

or $2a_{ii} = 0$ or $a_{ii} = 0$

Thus in a skew symmetric matrix all elements along the principal diagonal are zero.

Examples: $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, B = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

Note: A square matrix A is skew symmetric if and only if $A' = -A$.

(iii) ***Orthogonal matrix:*** A matrix A is said to be orthogonal If $A'A = I$ where A' is the transpose of A.

Some results related to symmetric and skew symmetric matrices:

(i) If A is any square matrix, then $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.

Proof: $(A + A')' = A' + (A')' = A' + A = A + A'$ $[\because (A + B)' = A' + B']$

Hence $A + A'$ is a symmetric matrix.

Again $(A - A')' = A' - (A')' = A' - A = -(A - A')$

Hence $A - A'$ is a skew symmetric matrix.

(ii) Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew symmetric matrix.

Proof: Let A be any square matrix. Then as in (i) $\frac{1}{2}(A + A')$ will be a symmetric matrix and $\frac{1}{2}(A - A')$ will be a skew symmetric matrix.

Let $B = \frac{1}{2}(A + A')$ and $C = \frac{1}{2}(A - A')$

Then $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = B + C$

where B is a symmetric matrix and C is a skew symmetric matrix.

To prove that the representation is unique: If possible let A = D + E where D is a symmetric and E is a skew symmetric matrix. Then

$$D' = D \text{ and } E' = -E \qquad \text{.....(1.20)}$$

Now $A = D + E \Rightarrow A' = D' + E' = D - E$ [From (1.20)]

Thus $A = D + E$ and $A' = D - E$ $\Big\}$ $\therefore$ $D = \frac{1}{2}(A + A') = B$ and $E = \frac{1}{2}(A - A') = C$

Hence representation of A is unique.

Determinant of a square matrix: Let A be a square of order n then the determinant of the matrix A is the value of the determinant whose elements are the corresponding elements of the matrix A and is denoted by |A| or Det. A

Thus if A = $[a_{ij}]$ be a square matrix of order n, then the number

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & an_2 & \cdots & a_{nn} \end{vmatrix}$$

is the determinant of the matrix A.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 3 & 4 & 5 \end{vmatrix} = 1(-5-8) - 0(10-12) + 3(4+3) = 8$

Minor and cofactor of an element of a determinant

Let A = $[a_{ij}]$ be a square matrix, then

(i) The minor of the element a_{ij} of |A| is the value of the determinant obtained by deleting its i^{th} row and j^{th} column and it is denoted by M_{ij}.

(ii) The cofactor of the element a_{ij} of |A| is denoted by the corresponding capital letter A_{ij} and $A_{ij} = (-1)^{i+j} M_{ij}$

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 5 \\ 0 & -3 & 6 \end{bmatrix}$

In det. A, minor of 1 = $\begin{vmatrix} 4 & 5 \\ -3 & 6 \end{vmatrix} = 24 + 15 = 39$

Cofactor of 1 = $(-1)^{1+1}$ 39 = 39 [$\because$ 1 lies in the 1^{st} row and 1^{st} column]

Minor of 2 = $\begin{vmatrix} -2 & 5 \\ 0 & 6 \end{vmatrix} = -12 - 0 = -12$

Cofactor of 2 = $(-1)^{1+2}$ (–12) = 12 [∵ 2 occurs in the 1st row and 2nd column]

Adjoint of a square matrix

Let A = $[a_{ij}]$ be a square matrix.

Let B = $[A_{ij}]$ where A_{ij} is the cofactor of the element a_{ij} in the det. A. The transpose B' of the matrix B is called the adjoint of the matrix A and is denoted by adj. A.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, then $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$

$$\text{adj. } A = B' = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$$

Theorem: If A is any square matrix of order n, then

A. (adj A) = (adj A). A = |A| I_n

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$

Then $\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{21} & A_{22} & \cdots & a_{n2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$

Now $A.\,(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$ [a square matrix of order n]

$$= |A|\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = |A| I_n$$

$$\because \sum_{j=1}^{n} a_{ij}A_{ij} = |A| \qquad \text{for } i = 1, 2, ..., n$$

And $$\sum_{j=1}^{n} a_{ij}A_{kj} = 0, \qquad i \neq k$$

Similarly we can show that (adj A). A = $|A|I_n$

Thus A(adj. A) = (adj. A) A = $|A|\ I_n$

Note: (i) $|A\ (\text{adj. } A)| = |A|^n$ $[\because\ |I_n| = 1]$

(ii) If $|A| \neq 0$, then $|\text{adj. } A| = |A|^{n-1}$

Non-singular and singular matrices

(i) A square matrix A is said to be a non-singular matrix if $|A| \neq 0$

(ii) A square matrix A is said to be a singular matrix if $|A| = 0$

Inverse or reciprocal of a square matrix: Let A be a square matrix of order n. Then a matrix B (if such a matrix exists) is called the inverse of A if

$$AB = BA = I_n$$

Inverse of the square matrix A is denoted by A^{-1}.

Existence of the inverse: The inverse of a square matrix A exists if and only if A is a non-singular matrix.

If part: Let A be non-singular square matrix of order n. Then $|A| \neq 0$.

Let $$B = \frac{\text{adj.}A}{|A|}$$

Then $$AB = \frac{A(\text{adj } A)}{|A|} = \frac{|A| I_n}{|A|} = I_n \qquad [\because A.\ (\text{adj } A) = |A|I_n] \qquad .....(1.21)$$

Hence B i.e., $\frac{\text{adj } A}{|A|}$ is the inverse of matrix A (by definition of inverse)

Only if part: Let A be a square matrix of order n. Let inverse of A exist. Let B be the inverse of A.

Then by definition of inverse

$AB = I_n \Rightarrow |AB| = |I_n| = 1$

or $|A|\,|B| = 1$ $[\because |AB| = |A|\,|B|]$

$\therefore$ $|A| \neq 0$, because product $|A|\,|B|$ is non-zero

Hence A is non singular

Note: (i) $A^{-1} = \dfrac{\text{adj } A}{|A|}$ (ii) $AA^{-1} = I_n$ [From (1.21)]

Theorem:

(i) If A and B be any two non-singular matrices, then AB is also a non-singular matrix and $(AB)^{-1} = B^{-1}A^{-1}$

$\because$ A, B are non-singular $\therefore |A| \neq 0, |B| \neq 0$

$|AB| = |A|\,|B| \neq 0$. Hence AB is non-singular

Non $AB(B^{-1}A^{-1})$

$= A\{B(B^{-1}A^{-1})\} = A\{(BB^{-1})A^{-1}\}$ [by associative law]

$= A\{I_n A^{-1}\}$ $[\because BB^{-1} = I_n]$

$= AA^{-1}$ $[\because I_n A^{-1} = A^{-1}]$

$= I_n$

Hence $B^{-1}A^{-1}$ is the inverse of AB

$(AB)^{-1} = B^{-1}A^{-1}$

(ii) If A is a non singular matrix, then $(A^{-1})^{-1} = A$

Let A be a square matrix of order n,

Then $A^{-1}A = I_n$ $\therefore$ inverse of $A^{-1} = A$ $\because (A^{-1})^{-1} = A$

$I_n^{-1} = I_n$ as $I_n^{-1} = I_n$

Elementary Operations or Elementary Transformations of a Matrix

Any of the following operations is called an elementary transformation.

(i) The interchange of any two rows (or columns)

(ii) The multiplication of the elements of any row (or column) by any non-zero number.

(iii) The addition to the elements of any row (or column) the corresponding elements of any other row (or column) multiplied by any number.

Note: An elementary transformation is called a row transformation or column transformation according as it applies to rows or columns.

1.3 Echelon Form of a Matrix

A matrix A is said to be in echelon form if

(i) every row of A which has all its elements 0, occurs below row which has a non zero element.

(ii) the first non-zero element in each non-zero row is 1

(iii) the number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Examples: (i) $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

***Note*:** A row of a matrix is said to be a zero row if all its elements are zero.

1.4 Rank of a Matrix

Definition: Let A be a matrix of order m × n. If atleast one of its minors of order r is different from zero and all minors of order (r +1) are zero, then the number r is called the rank of the matrix A and is denoted by $\rho(A)$.

***Note*:**

(i) The rank of a zero matrix is zero and the rank of an identity matrix of order n is n.

(ii) The rank of a non-singular matrix of order n is n.

(iii) The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

***Example* 1.1:** Reduce the following matrix to upper triangular form (Echelon form)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution: Upper triangular matrix. If in a square matrix, all the elements below the principal diagonal are zero, the matrix is called an upper triangular matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

Ans. $\overset{R_3 + 5\,R_2}{\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}}$

***Example* 1.2:** Transform $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ into a unit matrix

Solution:

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 0 & -1 & -5 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & -5 \end{bmatrix} -\frac{1}{2}R_2 \quad \sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \\ R_3 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} -\frac{1}{7}R_3 \quad \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 9R_3 \\ R_2 + 2R_3 \end{matrix}$$

Ans.

Elementary Matrices

A matrix obtained from a unit matrix by a single elementary transformation is called elementary matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider the matrix obtained by $R_2 + 3\ R_1$

$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called the elementary matrix

***Example* 1.3:** Use Gauss-Jordan reduction method (Elementary matrices method) to compute the inverse of the matrix.

$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ by applying elementary row transformation

Solution: $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

Elementary row transformation, which will reduce A = IA to I = PA, then matrix P will be in inverse of matrix A.

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A, \frac{1}{3}R_1 \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_2 - 2R_1 \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A - R_2 \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{matrix} R_1 + R_2 \\ R_3 + R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{2}{3} & -1 & 0 \\ \frac{2}{3} & -1 & 1 \end{bmatrix} A - 3R_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{2}{3} & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A$$

$$R_2 + \frac{4}{3}R_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A$$

$\therefore$ I = PA $\quad \therefore P = A^{-1}$ or $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$ **Ans.**

***Example* 1.4:** Compute the inverse of the following matrix by using elementary transformations.

$$\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} -1 & 4 & 1 & 2 \\ 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \quad \begin{matrix} R_1 \to R_3 \\ R_2 \to R_1 \\ R_3 \to R_2 \end{matrix}$$

$R_2 + 2R_1$, $R_3 + 5R_1$

$$\begin{bmatrix} -1 & 4 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 7 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$-R_1$, $\frac{1}{2} R_2$

$$\begin{bmatrix} 1 & -4 & -1 & -2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 7 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 - 7R_2$, $R_4 - R_2$

$$\begin{bmatrix} 1 & -4 & -1 & -2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -\frac{7}{2} & 1 & -2 & 0 \\ -\frac{1}{2} & 0 & -1 & 1 \end{bmatrix} A$$

$R_3 + R_4$, $R_2 - R_4$, $R_1 + 4R_4$

$$\begin{bmatrix} 1 & -4 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2 & 0 & -5 & 4 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -\frac{1}{2} & 0 & -1 & 1 \end{bmatrix} A$$

$2R_4$

$$\begin{bmatrix} 1 & -4 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -5 & 4 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

$R_1 + R_3$

$$\begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 1 & -8 & 5 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

$R_1 + 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

Inverse matrix $= \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}$ **Ans.**

***Example* 1.5:** Find the inverse of the matrix $A = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$\frac{1}{2}R_1$

$$\begin{bmatrix} 1 & 2 & 3/2 & 1 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A,$$

$R_2 - 3R_1, R_3, -2R_1, R_4 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 3/2 & 1 \\ 0 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & -1 & -5 \\ 0 & -3 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 3/2 & 1 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & \frac{1}{2} & -1 \\ 0 & -3 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -\frac{3}{2} & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} A,$$

$R_1 - 2R_2, R_4 + 3R_2, R_3 \to 2R_3$

$$\begin{bmatrix} 1 & 0 & \frac{7}{2} & 11 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 5 & -5 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 0 & -2 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 2 & 0 & 0 \\ -5 & 0 & 3 & 1 \end{bmatrix} A$$

$R_1 - \frac{7}{2} R_3, R_2 + R_3, R_4 - 5R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 13 & -7 & -2 & 0 \\ -4 & 2 & 1 & 0 \\ -3 & 2 & 0 & 0 \\ 10 & -10 & 3 & 1 \end{bmatrix} A,$$

$\frac{1}{5} R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -7 & -2 & 0 \\ -4 & 2 & 1 & 0 \\ -3 & 2 & 0 & 0 \\ 2 & -2 & \frac{3}{5} & \frac{1}{5} \end{bmatrix} A$$

$R_1 - 18R_4,\ R_2 + 7R_4,\ R_3 + 2R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -23 & 29 & -\frac{64}{5} & -\frac{18}{5} \\ 10 & -12 & \frac{26}{5} & \frac{7}{5} \\ 1 & -2 & \frac{6}{5} & \frac{2}{5} \\ 2 & -2 & \frac{3}{5} & \frac{1}{5} \end{bmatrix} A,$$

$$A^{-1} = \begin{bmatrix} -23 & 29 & -\frac{64}{5} & -\frac{18}{5} \\ 10 & -12 & \frac{26}{5} & \frac{7}{5} \\ 1 & -2 & \frac{6}{5} & \frac{2}{5} \\ 2 & -2 & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$$ **Ans.**

Exercise 1.1

Find the inverse of the following matrices:

1. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -1 & -1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ **Ans.** $\frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 6 & 8 \end{bmatrix}$ **Ans.** $\frac{1}{20}\begin{bmatrix} -4 & -18 & 10 \\ 10 & 25 & -15 \\ -6 & -12 & 10 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ **Ans.** $\frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 1 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}$ **Ans.** $\frac{1}{18}\begin{bmatrix} 2 & 5 & -7 & 1 \\ 5 & -1 & 5 & -2 \\ -7 & 5 & 11 & 10 \\ 1 & -2 & 10 & 5 \end{bmatrix}$

***Example* 1.6:** Determine the rank of a matrix

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$$

Solution :

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & -5 & 7 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & 7 \end{bmatrix} -\frac{1}{2}R_2 \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{bmatrix} R_3 \times 5R_2$$

Rank = Number of non-zero rows = 3 **Ans.**

***Example* 1.7:** Find the rank of the matrix

$$\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix} \begin{matrix} R_2 + 2R_1 \\ R_3 + 3R_1 \\ R_4 + 5R_1 \end{matrix}$$

$$\sim\begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_3 - 2R_2 \\ R_4 - 2R_2 \\ \\ \end{matrix}$$

Here the 4th order and 3rd order minors are zero. But a minor of second order

$$\begin{vmatrix} 3 & -2 \\ 7 & -2 \end{vmatrix} = -6 + 14 = 8 \neq 0$$

Rank = Number of non-zero rows = 2 **Ans.**

***Example* 1.8:** Find the rank of the matrix

$$\begin{bmatrix} 3 & -4 & -1 & 2 \\ 1 & 7 & 3 & 1 \\ 5 & -2 & 5 & 4 \\ 9 & -3 & 7 & 7 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 7 & 3 & 1 \\ 3 & -4 & -1 & 2 \\ 5 & -2 & 5 & 4 \\ 9 & -3 & 7 & 7 \end{bmatrix} R_1 \leftrightarrow R_2$$

$R_2 - 3R_1$, $R_3 - 5R_1$, $R_4 - 9R_1$, $R_3 - \frac{37}{25}R_2$, $R_4 - \frac{66}{25}R_2$ $R_4 - \frac{4}{3}R_3$

$$\begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & -37 & -10 & -1 \\ 0 & -66 & -20 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & 0 & \frac{24}{5} & \frac{12}{25} \\ 0 & 0 & \frac{32}{5} & \frac{16}{25} \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & 0 & \frac{24}{5} & \frac{12}{25} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank = 3 **Ans.**

***Example* 1.9:** Find the rank of

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution:

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & -7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \quad R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \end{bmatrix} \quad R_3 - \frac{4}{5}R_2, R_4 - \frac{9}{5}R_2 \quad - \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 - R_3$$

Rank = Number of non-zero rows = 3 **Ans.**

Exercise 1.2

Find the rank of the following matrices:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$ **Ans.** 2

2. $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ **Ans.** 2

3. $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$ **Ans.** 3

4. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ **Ans.** 3

5. $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ **Ans.** 2

6. $\begin{bmatrix} 2 & 4 & 3 & -2 \\ -3 & -2 & -1 & 4 \\ 6 & -1 & 7 & 2 \end{bmatrix}$ **Ans.** 3

7. $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ **Ans.** 4

8. $\begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$ **Ans.** 4

1.5 Normal Form or Canonical Form

(i) A matrix of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where I_r is a unit matrix of order 'r' and 0 is the null matrix is called the normal form or first canonical form.

(ii) Matrices of the form $[I_r \quad 0]$, $[I_r]$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, are also known as normal forms.

(iii) Every m × n matrix can be reduced to the normal form by a series of elementary transformations.

(iv) The rank of the above canonical form is 'r'.

(v) *Algorithm to reduce a matrix to the normal form :*

***Step* 1 :** Make $a_{11} = 1$, by applying suitable row or column operations or both.

***Step* 2 :** Make all the elements below a_{11} in 1st column and on the right of a_{11}, in the 1st row equal to zero with the help of suitable row and column operations.

***Step* 3 :** If the resulting matrix is in the normal form, the process ends. Otherwise repeat the above steps regarding a_{22} such that no row or column operation should involve 1st row or 1st column. If the matrix is still not in the normal form proceed to a_{33} and apply the same process. The process is continued till the normal form is obtained.

***Example* 1.10:** Reduce the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 & 6 \\ 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 8 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$ to an Echelon form and hence find its rank.

Solution:

Step 1 : To get '1' in the first position of 1st row, (i.e., a_{11} position).

Applying R14 ($R_1 \leftrightarrow R_4$), we get

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 8 \\ 3 & 1 & 4 & 6 \end{bmatrix}$$

Step 2 : To get zeros below a_{11}.

Applying $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 3R_1$,

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

Step 3 : To get 1 in a_{22} position.

Applying $R_2 \to -R_2$,

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

Step 4 : To get zeros below a_{22} :

Applying $R_3 \to R_3 + 2R_2$, $R_4 \to R_4 + 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Step 5 : $a_{33} = 1$; so we have to get zeros below it. $\therefore$ Applying $R_4 \to R_4 - 2R_3$;

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is in the Echelon form which has 3 non-zero rows in it.

$\therefore \qquad \rho(A) = 3$

***Example* 1.11:** Apply elementary transformations to find the rank of

$$A = \begin{bmatrix} 1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix}$$

Solution: Applying $R_2 \to R_2 - 7R_1$; $R_3 \to R_3 - 5R_1$, we get,

$$A \sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 69 & -23 & 46 \\ 0 & 33 & -11 & 22 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 3 & -1 & 2 \\ 0 & 3 & -1 & 2 \end{bmatrix} \qquad \left(R_2 \to \frac{1}{23}R_2; R_3 \to \frac{1}{11}R_3\right)$$

$$\sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B \text{ (say) } [R_3 \to R_3 - R_2],$$

which is in Echelon form which has two non-zero rows,

$\therefore \qquad \rho(A) = \rho(B) = 2$

(or) every 3-rowed minor of B = 0

and $\qquad \begin{vmatrix} 1 & -7 \\ 0 & 3 \end{vmatrix} \neq 0.$

$\therefore \qquad \rho(A) = 2$

***Example* 1.12:** Find the rank of the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$,

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad \left\{\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix}\right\}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(R_4 \to R_4 - (R_2 + R_3)\right)$$

= B ; In B, the 4 rowed minor = 0

$\therefore \quad \rho(A) \neq 4$

The leading 3 rowed minor,

$$= \begin{vmatrix} 1 & -1 & -2 \\ 0 & 5 & 3 \\ 0 & 4 & 9 \end{vmatrix} = 45 - 12 \neq 0$$

$\therefore \quad \rho(A) = 3.$

(**Note :** The problem can also be solved by reducing to Echelon form).

***Example* 1.13:** Find the constants 'l' and 'm' such that the rank of the matrix $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & l & m \end{bmatrix}$ is (i) 3 (ii) 2.

Solution:

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & l & m \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 10 & (l-18) & (m-6) \end{bmatrix} \begin{Bmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 6R_1 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & (l-4) & (m-6) \end{bmatrix} \left(R_3 \to R_3 - 2R_2\right) = B$$

Matrix B is in Echelon form.

(i) If $\rho(A) = 3$, then $\rho(B) = 3$, $\therefore$ B should have 3 non zero rows.

$\therefore$ $l \neq 4$ (or) $m \neq 6$.

(ii) If $\rho(A) = 2$, then $\rho(B) = 2$, $\therefore$ B should have 2 non-zero rows.

$\therefore$ $l = 4$ and $m = 6$.

***Example* 1.14:** Find the rank of the matrix $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 3 & 4 & 8 & 11 \\ 1 & 3 & 6 & 7 \end{bmatrix}$.

Solution: The given matrix be A. Then,

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 3 & 4 & 8 & 11 \\ 1 & 3 & 6 & 7 \end{bmatrix} (R_1 \leftrightarrow R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 4 & 4 \end{bmatrix}, \begin{Bmatrix} R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix} = B \text{ (say)}$$

B is in Echelon form with two non zero rows.

$\therefore$ $\rho(A) = \rho(B) = 2$.

***Example* 1.15:** Reduce the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 3 & 4 & 5 \end{bmatrix}$ to normal form and hence find its rank.

Solution: Given matrix be A. Then,

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix}, \begin{Bmatrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix}, \begin{Bmatrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 3C_1 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{Bmatrix} R_2 \to R_2 \times -\frac{1}{2} \\ R_3 \to R_3 - R_2 \end{Bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, (C_3 \to C_3 - 2C_2),$$

which is of the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence $\rho(A) = 2$.

***Example* 1.16:** Reduce the matrix $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \end{bmatrix}$ into the normal form and hence find its rank.

Solution: Given matrix be A. Then

$$A \sim \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 2 & -2 & 2 \\ 0 & 10 & -4 & -6 \\ 0 & 1 & -1 & 1 \end{bmatrix}, \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 10 & -4 & -6 \\ 0 & 1 & -1 & 1 \end{bmatrix}, \begin{matrix} C_2 \to C_2 + 2C_1 \\ C_3 \to C_3 - C_1 \\ C_4 \to C_4 - 2C_1 \end{matrix}.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 10 & -4 & -6 \\ 0 & 1 & -1 & 1 \end{bmatrix} R_2 \to \frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 6 & -6 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} C_2 \to C_2 + C_3 + C_4 \\ C_3 \to C_3 + C_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} R_3 \to \frac{1}{6} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} C_4 \to C_4 - C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} C_4 \to C_4 + C_3$$

which is of the normal form $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $\rho(A) = 3$.

***Example* 1.17:** Reduce the matrix $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 1 & 3 & 2 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$, to normal form and find its rank.

Solution: Given matrix be A. Then

$$A \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 3 & 5 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix}, \begin{matrix} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&4&5\\0&3&5&-2\\0&1&1&-2\end{bmatrix},\ \begin{matrix}C_3\to C_3+3C_1\\C_4\to C_4-2C_1\end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&4&5\\0&0&-7&-17\\0&0&-3&-7\end{bmatrix},\ \begin{matrix}R_3\to R_3-3R_2\\R_4\to R_4-R_2\end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-7&-17\\0&0&-3&-7\end{bmatrix},\ \begin{matrix}C_3\to C_3-4C_2\\C_4\to C_4-5C_2\end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&\frac{17}{7}\\0&0&-3&-7\end{bmatrix},\ R_3\to\frac{-1}{7}R_3$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&\frac{17}{7}\\0&0&0&\frac{2}{7}\end{bmatrix},\ R_4\to R_4+3R_3$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{2}{7}\end{bmatrix},\ C_4\to C_4-\frac{17}{7}C_3$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ R_4\to\frac{7}{2}R_4$$

which is in the normal form I_4. $\therefore\ \rho(A)=4$

***Example* 1.18:** If $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix}$, find two non-singular matrices P and Q such that PAQ is in the normal form.

Method of finding two non-singular matrices P and Q such that PAQ is in the normal form where A is any given m × n matrix.

Step 1 :

Write $A = I_m A I_n,$ .....(1.22)

where I_m, I_n are unit matrices of orders m and n respectively.

Step 2 :

Subject the matrix A on the L.H.S of (1.22) to elementary row and column transformations to reduce it to the normal form. Perform each of the same row transformations on I_m of R.H.S and each of the same column transformations on I_n of R.H.S. In all the steps keep the matrix A on R.H.S as it is.

Solution: Given matrix be A. Since it is of order 3 × 4, we write

$$A = I_3 A I_4$$

i.e.,
$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 2R_1$ on L.H.S as well as the prefactor (I_3) of A on R.H.S, we get,

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 6 & 6 & -9 \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 + C_1$, $C_3 \to C_3 + C_1$, $C_4 \to C_4 - 2C_1$, on L.H.S as well as the postfactor (I_4) of A on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 6 & -9 \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to \frac{1}{6} R_2$ on L.H.S as well as the prefactor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{-3}{2} \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{6} & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 4R_2$ on L.H.S as well as prefactor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{-3}{2} \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{6} & 0 \\ \frac{2}{3} & \frac{-2}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_3 \to C_3 - C_2$ on L.H.S as well as the postfactor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{2} \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{6} & 0 \\ \frac{2}{3} & \frac{-2}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to \frac{-1}{2} R_3$ on L.H.S as well as the prefactor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{2} & \frac{1}{6} & 0 \\ \frac{-1}{3} & \frac{1}{3} & \frac{-1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_4 \to C_4 + \left(\frac{3}{2}\right) C_2$ on L.H.S as well as postfactor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{6} & 0 \\ \frac{-1}{3} & \frac{1}{3} & \frac{-1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & \frac{-1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The L.H.S which is now of the form $[I_3\ 0]$ is in the normal form. Hence the two required matrices are

$$P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{6} & 0 \\ \frac{-1}{3} & \frac{1}{3} & \frac{-1}{2} \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 1 & 0 & \frac{-1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

***Example* 1.19:** Find two non-singular matrices P and Q such that P AQ will be in the normal form where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution: Here A is 3×3 matrix, hence we write $A = I_3 A I_3$

i.e.,
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ on L.H.S as well as the prefactor of A on R.H.S, we get,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -6 \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$ on L.H.S as well as the postfactor on R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & -6 \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to \frac{-1}{5} R_2$ on L.H.S as well as the prefactor of R.H.S, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{6}{5} \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{-1}{5} & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 + 5R_2$ on L.H.S and the prefactor on R.H.S, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{6}{5} \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{-1}{5} & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_3 \to C_3 - \frac{6}{5}C_2$, on L.H.S and the postfactor on R.H.S, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{-1}{5} & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & \frac{-3}{5} \\ 0 & 1 & \frac{-6}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to -R_3$ on L.H.S and the prefactor of R.H.S, we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{-1}{5} & 0 \\ 1 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & \frac{-3}{5} \\ 0 & 1 & \frac{-6}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Now the L.H.S is in the normal form (I_3). Hence the required matrices are

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 2/5 & -1/5 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -2 & \frac{-3}{5} \\ 0 & 1 & \frac{-6}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise - 1.3

I. Reduce the following matrices to Echelon form and hence find their ranks.

(a) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ **[Ans : 3]**

(b) $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & -3 & 1 & 4 \\ 3 & 4 & -2 & 6 \end{bmatrix}$ **[Ans : 3]**

(c) $\begin{bmatrix} 3 & 4 & 5 & 2 \\ -1 & 2 & 0 & -3 \\ 1 & -1 & 1 & 4 \end{bmatrix}$ **[Ans : 3]**

(d) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ **[Ans : 2]**

II Reduce the following matrices to the normal form and hence find their ranks.

(a) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **[Ans : 2]**

(b) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}$ **[Ans : 2]**

(c) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ **[Ans : 2]**

(d) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & -4 \end{bmatrix}$ **[Ans : 3]**

III Find two non-singular matrices P and Q such that P AQ is in the normal form where A is given by the matrix.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$ **Ans :** $P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 3 & -2 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ **Ans :** $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{-1}{3} & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

IV Find the constants 'a' and 'b' such that the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 2 & -3 \\ 1 & 1 & a & b \end{bmatrix}$ **is of rank**

(i) 3 and (ii) 2.

Ans : (i) $a \neq \frac{1}{2}$ **(or)** $b \neq \frac{7}{2}$

(ii) $a = \frac{1}{2}$ and $b = \frac{7}{2}$

V If the rank of the matrix $\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -1 \\ 3 & 0 & a \end{bmatrix}$ **is '2', find the value of 'a'.**

[Ans : 2]

1.6 Simultaneous Linear Equations

In this section, the theory of matrices is applied to determine the existence of solutions of simultaneous linear equations and obtain the solutions if they exist.

Matrix form of a system of non-homogeneous Linear Equations

A system of 'm' simultaneous non-homogeneous linear equations in 'n' unknowns $x_1, x_2, x_3, ...x_n$ is of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots a_{mn}x_n = b_m \end{array}\right\} \quad .....(1.23)$$

The above system can be written in a matrix form (using the concepts of multiplication of matrices and equality of matrices) as

$$\begin{bmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2n} \\ \cdots & \cdots & ..... & \cdots \\ a_{m1} & a_{m2} & ..... & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad .....(1.24)$$

or simply, $AX = B$ .....(1.25)

where $B \neq 0$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$, which is called the coefficient matrix,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$ which is the solution matrix

and $$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$ is the matrix of the constants b_1, b_2, ... b_m of the R.H.S of (1.23)

The set of values x_1, x_2,...x_n which satisfy the system (1.23) is called the solution of the system.

Consistency and Inconsistency

The system of equations (1.23) is said to be consistent if it has at least one solution.

(1.23) is said to be inconsistent if it has no solution.

Augmented matrix (A/B)

The $m \times (n + 1)$ matrix given by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

(which is obtained by annexing the elements of B to those of (A) is called the augmented matrix and is denoted by (A/B) or (A, B).

Condition for the consistency of (1)

1. The condition for the consistency of (1) is given by the following theorem (with out proof).
2. ***Theorem*** **:** The system of simultaneous linear equations AX = B is consistent if and only if the rank of A = rank of (A/B).

Note: 1. Thus the ranks of the coefficient matrix and the augmented matrix play a vital role in determining the consistency of the system (1).

2. We apply the elementary transformations to the augmented matrix (A/B) since they do not alter its rank. This reduced equivalent system will enable us to test for consistency and also to find the solution.

Working Rules

Step 1 : Write the system in the form AX = B.

Step 2 : Find $\rho(A)$ and $\rho(A/B)$ by applying elementary transformations on (A/B).

Conclusions :

Case (i) : If $\rho(A) = \rho(A/B) = n$, where 'n' is the number of unknowns, the system is consistent and it has a unique solution [in this case A is non-singular]. This unique solution (given by $X = A^{-1} B$) can be obtained by the reduced system of equations.

Case (ii) : If $\rho(A) = \rho(A/B) < n$, then the system is consistent and has an infinite number of solutions.

Case (iii) : If $\rho(A) \neq \rho(A/B)$, the system is inconsistent and has no solution.

Note : $\rho(A) \ngtr \rho(A/B)$

Summary

The nature of solutions of the system AX = B with respect to its consistency or inconsistency is shown in the following table.

Consistent	$\rho(A, B) = \rho(A) = n$	unique solution
	$\rho(A, B) = \rho(A) < n$	Infinite solutions
Inconsistent	$\rho(A, B) \neq \rho(A)$	No solution
	i.e, $\rho(A, B) > \rho(A)$	

Some other methods of solving non-homogenous linear equations

In this section, certain methods of solving non-homogeneous equations (in which the concept of the rank of a matrix need not be used) are discussed.

Note that these are applicable only when the coefficient matrix of the system is a square matrix.

Method 1 : *Matrix inversion method :*

The solution of the system AX = B is given by $X = A^{-1}B$ provided A^{-1} exists.

∴ This method is applicable only when A is nonsingular.

Method 2 : *CRAMER'S RULE (using determinants) :*

To apply this method A must be non-singular.

i.e., $|A| \neq 0$

Let A be 3 × 3 matrix.

Let $|A| = \Delta$

we find the values of 3 other determinants which are obtained by replacing the 1st, 2nd and 3rd columns of Δ with the column matrix 'B' of the given system respectively.

Let $\Delta_1, \Delta_2, \Delta_3$ be the values of these determinants respectively.

Then, $$x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta},\ z = \frac{\Delta_3}{\Delta}$$

Method 3 : *Gauss-Jordan method:*

Take the matrix (A, B) and subject it to a series of elementary row transformations till it is reduced to the form $(I_n X)$ where I_n is unit matrix of order n and X is the column matrix of the order of B. Then 'X' is the solution matrix.

Homogeneous Linear equations

The system of equations AX = 0 is said to be homogeneous where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad \text{and } 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$$

Here the augmented matrix is

$$(A/B) = (A/0)$$

Hence $\rho(A/B) = \rho(A)$

Conclusions :

∴ 1. The system AX = 0 is always consistent. [Obviously, the trivial solution $x_1 = x_2 = ... = x_n = 0$ always exists].

2. If $\rho(A/B) = \rho(A) = n$ (i.e., is $|A| \neq 0$) then the trival solution is the unique solution.

3. If $\rho(A/B) = \rho(A) = r < n$, the system has non-trivial solutions. (In this case $|A| = 0$).

Let $\rho(A/B) = \rho(A) = r < n$. Then the system will have $(n - r)$ linearly independent solutions. The values of 'r' unknowns can be expressed in these arbitrarily chosen $(n - r)$ unknowns.

Solved Examples

***Example* 1.20:** Test the system of equations

$$2x + y + 5z = 4;\ 3x - 2y + 2z = 2\ ;\ 5x - 8y - 4z = 1$$

for consistency. If consistent solve them.

Solution: The given system of equations can be put in the matrix form AX = B where,

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & -2 & 2 \\ 5 & -8 & -4 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

Augmented matrix

$$(A/B) = \begin{bmatrix} 2 & 1 & 5 & 4 \\ 3 & -2 & 2 & 2 \\ 5 & -8 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} & 2 \\ 3 & -2 & 2 & 2 \\ 5 & -8 & -4 & 1 \end{bmatrix},\ R_1 \rightarrow \frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} & 2 \\ 0 & \frac{-7}{2} & \frac{-11}{2} & -4 \\ 0 & \frac{-21}{2} & \frac{-33}{2} & -9 \end{bmatrix},\ \begin{matrix} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} & 2 \\ 0 & 1 & \frac{11}{7} & \frac{8}{7} \\ 0 & \frac{-21}{2} & \frac{-33}{2} & -9 \end{bmatrix} R_2 \to \frac{-2}{7} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{12}{7} & \frac{10}{7} \\ 0 & 1 & \frac{11}{7} & \frac{8}{7} \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{1}{2} R_2 \\ R_3 \to R_3 + \frac{21}{2} R_2 \end{matrix}$$

From the above equivalent matrix, we find that rank of A = 2; rank of (A/B) = 3

$\therefore$ $\rho(A) \neq \rho(A/B)$

Hence the system is not consistent

$\therefore$ It has no solution.

***Example* 1.21:** Solve the system $2x - y + 4z = 12$; $3x + 2y + z = 10$; $x + y + z = 6$; if it is consistent.

Solution: The matrix form of the system is

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 6 \end{bmatrix} \quad .....(1.26)$$

or $AX = B$

where $A = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 12 \\ 10 \\ 6 \end{bmatrix}$

Augmented matrix

$$(A, B) = \begin{bmatrix} 2 & -1 & 4 & 12 \\ 3 & 2 & 1 & 10 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 2 & 1 & 10 \\ 2 & -1 & 4 & 12 \end{bmatrix}, R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -2 & -8 \\ 0 & -3 & 2 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & -3 & 2 & 0 \end{bmatrix} R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 8 & 24 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 3R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} R_3 \to \frac{1}{8} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 2R_3 \end{matrix} \qquad \text{.....(1.27)}$$

which is in Echelon form.

$$\rho(A, B) = \rho(A) = 3$$

So the system is consistent and has unique solution.

Further (1.27) is equivalent to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \qquad x = 1 \; ; y = 2 \; ; z = 3$$

***Example* 1.22:** If consistent, solve the system of equations,

$$x + y + z + t = 4$$
$$x - z + 2t = 2$$
$$y + z - 3t = -1$$
$$x + 2y - z + t = 3$$

Solution: The system can be written as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 1 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 3 \end{bmatrix} \qquad \text{.....(1.28)}$$

The augmented matrix

$$(A, B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 1 & 2 & -1 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & 1 & -2 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 1 & -2 & 0 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 1 & -2 & 0 & -1 \end{bmatrix} R_2 \to -R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & -4 & 1 & -3 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & 1 & -3 \end{bmatrix} R_3 \to -R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & 1 & 0 & -5 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 9 & 9 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 2R_3 \\ R_4 \to R_4 + 4R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & 1 & 0 & -5 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} R_4 \to \frac{1}{9} R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 4R_4 \\ R_2 \to R_2 + 5R_4 \\ R_3 \to R_3 - 2R_4 \end{matrix} \qquad .....(1.29)$$

which is in Echelon form

$$\rho(A) = \rho(A/B) = 4 = \text{number of unknowns}$$

∴ The system (1.28) is consistent and has a unique solution.

Again from the reduced form (1.29) of (A/B), the system becomes,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

⇒ $x = 1,\ y = 1,$
$z = 1, \text{ and } t = 1;$

***Example* 1.23:** Test for consistency and if consistent solve the system,

$$5x + 3y + 7t = 4$$

$$3x + 26y + 2t = 9$$

$$7x + 2y + 10t = 5$$

Solution: The given system can be written as

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} \qquad .....(1.30)$$

or $AX = B,$

where $A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

The augmented matrix

$$(A, B) = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$$\sim\begin{bmatrix}5 & 3 & 7 & 4\\ 3 & 26 & 2 & 9\\ 2 & -1 & 3 & 1\end{bmatrix} R_3 \to R_3 - R_1$$

$$\sim\begin{bmatrix}5 & 3 & 7 & 4\\ 1 & 27 & -1 & 8\\ 2 & -1 & 3 & 1\end{bmatrix} R_2 \to R_2 - R_3$$

$$\sim\begin{bmatrix}1 & 27 & -1 & 8\\ 5 & 3 & 7 & 4\\ 2 & -1 & 3 & 1\end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim\begin{bmatrix}1 & 27 & -1 & 8\\ 0 & -132 & 12 & -36\\ 0 & -55 & 5 & -15\end{bmatrix} \begin{matrix}R_2 \to R_2 - 5R_1\\ R_3 \to R_3 - 2R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 27 & -1 & 8\\ 0 & 1 & -\frac{1}{11} & \frac{3}{11}\\ 0 & -55 & 5 & -15\end{bmatrix} R_2 \to \frac{-1}{132}R_2$$

$$\sim\begin{bmatrix}1 & 0 & 16/11 & 7/11\\ 0 & 1 & -1/11 & 3/11\\ 0 & 0 & 0 & 0\end{bmatrix} \begin{matrix}R_1 \to R_1 - 27R_2\\ R_3 \to R_3 + 55R_2\end{matrix} \qquad .....(1.31)$$

which is in Echelon form.

$\therefore$ $\rho(A, B) = \rho(A) = 2 <$ the number of unknowns.

$\therefore$ The system (1.30) is consistent with infinite number of solutions.

$\rho(A) = r = 2$, number of unknowns $n = 3$.

Hence we have to consider $(3 - 2 = 1)$ one variable as an arbitrary constant and the other two variables are expressed in terms of this constant.

The final reduced system (1.31) can be written as

$$\left.\begin{aligned} x + \frac{16t}{11} &= \frac{7}{11}\\ y - \frac{t}{11} &= \frac{3}{11}\end{aligned}\right\} \qquad .....(1.32)$$

$$\Rightarrow \qquad y = \frac{3+t}{11};\ x = \frac{7-16t}{11}$$

If t = c, where c is an arbitrary constant, we have, the solution of the system (1.30) as,

$$x = \frac{1}{11}(7 - 16c),\ y = \frac{1}{11}\ (c+3),$$

For different values of '*c*' we get different solutions. Thus we get infinite number of solutions.

***Example* 1.24:** Solve the system $x - z - 2 = 0$; $y - z + 2 = 0$; $x + y - 2z = 0$; if it is consistent.

***Solution*:** The system is AX = B, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

Augmented matrix

$$(A, B) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & -2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{bmatrix} R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

$\rho(A, B) = \rho(A) = 2 <$ number of unknowns 3.

So the system being consistent, will have infinite number of solutions which involve $3 - 2 = 1$ arbitrary constant.

The final reduced matrix ~ (A,B) makes the system equivalent to,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$\Rightarrow \quad x - z = 2,\ y - z = -2$

$\Rightarrow \quad x = z + 2,\ y = z - 2$

Taking z = c (an arbitrary constant) we get the solution as,

$$x = c + 2,\ y = c - 2,\ z = c.$$

***Example* 1.25:** Investigate for what values of α, β, the system of equations given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \alpha \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \beta \end{bmatrix}, \text{ has}$$

(i) no solution (ii) unique solution (iii) an infinite number of solutions.

***Solution*:** The given system can be put in the form

$$AX = B, \text{ where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \alpha \end{bmatrix}, X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, B = \begin{bmatrix} 6 \\ 10 \\ \beta \end{bmatrix}$$

Augmented matrix

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \alpha & \beta \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & (\alpha - 1) & (\beta - 6) \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & (\alpha - 3) & (\beta - 10) \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$= C \text{ (say)}$$

Case (i) :

If the system has no solution,

$$\rho(A/B) \neq \rho(A)$$

i.e., $\quad \alpha = 3,\ \beta \neq 10$

Explanation :

If $\alpha = 3$; $\beta \neq 10$;

$$A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \rho(A) = 2 \text{ and } (A, B) \sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & \gamma \end{bmatrix}$$

(where $\gamma = \beta - 10 \neq 0$)

$\therefore$ $$\begin{vmatrix} 0 & -1 & 2 \\ 1 & 2 & 4 \\ 0 & 0 & \gamma \end{vmatrix} \neq 0, \text{ so } \rho(A,B) = 3$$

Hence $\rho(A) \neq \rho(A, B)$

$\therefore$ System has no solution.

Case (ii) :

If the system has a unique solution, $\rho(A) = \rho(A,B)$ = number of unknowns = 3

$\therefore$ $\alpha - 3 \neq 0$, and $(\beta - 10)$ can take any value

i.e., $\alpha \neq 3$, and β can have any value

Explanation : $$\alpha \neq 3 \Rightarrow C = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \delta & \beta-10 \end{bmatrix}$$ (where $\delta = \alpha - 3 \neq 0$)

$\therefore$ $$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & \delta \end{vmatrix} = \delta \neq 0$$

$\therefore$ $\rho(A) = \rho(A, B) = 3$

Case (iii) :

The system has infinite number of solutions (say).

Then $\rho(A) = \rho(A, B) < 3$ number of unknowns.

$\therefore$ If $\alpha = 3$, and $\beta = 10$, $$C = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$, so that $\rho(A) = \rho(A, B) = 2 < 3$

$\therefore$ $\alpha = 3$ and $\beta = 10$.

Hence the required answers are

(i) $\alpha = 3, \beta \neq 10$

(ii) $\alpha \neq 3$, β can have any value

(iii) $\alpha = 3$ and $\beta = 10$.

***Example* 1.26:** Test the following system for consistency and if consistent solve it.

$$u + 2v + 2w = 1,\ 2u + v + w = 2,\ 3u + 2v + 2w = 3,\ v + w = 0.$$

Solution: The system is in the form AX = B, where

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

Augmented matrix

$$(A,B) = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -5 & -5 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 \to \frac{-1}{5} R_2 \\ R_3 \to \frac{-1}{4} R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

Which is in Echelon form with two nonzero rows.

$\therefore$ $\rho(A) = \rho(A, B) = 2 < 3$ (number of unknowns)

$\therefore$ The system is consistent and has infinite solutions containing $3 - 2 = 1$ arbitrary constant 'c' say.

The system is reduced to

$$\left.\begin{array}{l} u + 2v + 2w = 1 \\ v + w = 0 \end{array}\right\} \text{[from final equivalent matrix]}$$

Let $w = c$, then $v = -c$

and $u = 1$ ($\because$ $2v + 2w = 0$)

$$\therefore \quad X = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -c \\ c \end{bmatrix},$$

where c is arbitrary, is the solution.

***Example* 1.27:** Solve the following system completely.

$x + y + z = 1$; $x + 2y + 4z = \alpha$

$x + 4y + 10z = \alpha^2$

***Solution*:** The system can be put in the matrix form AX = B, as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix} \quad .....(1.33)$$

Here $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}$

Augmented matrix

$$(A,B) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \alpha \\ 1 & 4 & 10 & \alpha^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \alpha - 1 \\ 0 & 3 & 9 & \alpha^2 - 1 \end{bmatrix} R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 2-\alpha \\ 0 & 1 & 3 & \alpha-1 \\ 0 & 0 & 0 & \alpha^2-3\alpha+2 \end{bmatrix} R_1 \to R_1 - R_2,\ R_3 \to R_3 - 3R_2$$

$$= C \text{ (say)}$$

For the system to be consistent, $\rho(A)$ and $\rho(A, B)$ must be equal.

$\because$ $$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix},\ \rho(A) = 2$$

$\therefore$ $\rho(A, B) = \rho(C) = 2 \Rightarrow \alpha^2 - 3\alpha + 2 = 0 \Rightarrow \alpha = 1$ or $\alpha = 2$.

$\therefore$ The system is consistent when $\alpha = 1$ or 2.

Case (i) :

$$\alpha = 1 \Rightarrow C = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x - 2z = 1,\ y + 3z = 0$ .....(1.34)

$\because$ $\rho(A) = \rho(A, B) = 2 <$ number of unknowns 3, the system has infinite solutions involving one arbitrary constant.

$\therefore$ Let $z = c_1$, so that (1.34) gives $y = -3z$

$\Rightarrow$ $y = -3c_1$ and $x = 1 + 2z$

$\Rightarrow$ $x = 1 + 2c_1$

$\therefore$ when $\alpha = 1$, the solution is,

$x = 1 + c_1$,
$y = -3c_1$,
$z = c_1$,

where c_1 is an arbitrary constant.

Case (ii) :

$$\alpha = 2 \Rightarrow C = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \qquad x - 2z = 0,\ y + 3z = 1,$

Taking $z = c_2$ an arbitrary constant, we get the solution as,

$x = 2c_2,$
$y = 1 - 3c_2,$
$z = c_2$

***Example* 1.28:** Show that the system

$$3x + 4y + 5z = \alpha,$$
$$4x + 5y + 6z = \beta,$$
$$5x + 6y + 7z = \gamma,$$

is consistent only when α, β, γ are in A . P.

***Solution*:** The given system, when put in matrix form is

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Which when compared with AX = B gives

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Augmented matrix

$$(A, B) = \begin{bmatrix} 3 & 4 & 5 & \alpha \\ 4 & 5 & 6 & \beta \\ 5 & 6 & 7 & \gamma \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \beta-\alpha \\ 1 & 1 & 1 & \gamma-\beta \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \beta-\alpha \\ 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \gamma-\beta \end{bmatrix}, R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \beta-\alpha \\ 0 & 1 & 2 & 4\alpha-3\beta \\ 0 & 0 & 0 & \gamma-2\beta+\alpha \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$\Rightarrow$ $\rho(A) = 2$ for any α, β, γ, and ρ (A, B) = 2 when $\alpha - 2\beta + \gamma = 0$.

But the system is consistent if and only if

$$\rho(A) = \rho(A, B)$$

i.e., if $\rho(A, B) = 2$,

i.e., if $\alpha - 2\beta + \gamma = 0$

i.e., if $\alpha + \gamma = 2\beta$,

i.e., when α, β, γ are in A.P.

***Example* 1.29:** Find the totality of solutions of the system given by,

$$5x + 2y - 6z + 2t + 1 = 0$$
$$x - y + z - t + 2 = 0$$

***Solution*:** The given system is,

$$\begin{bmatrix} 5 & 2 & -6 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Which is in the form AX = B, where

$$A = \begin{bmatrix} 5 & 2 & -6 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}, B = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Augmented matrix *X*,

$$(A, B) = \begin{bmatrix} 5 & 2 & -6 & 2 & -1 \\ 1 & -1 & 1 & -1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 2 & -6 & 2 & -1 \\ 0 & \frac{-7}{5} & \frac{11}{5} & \frac{-7}{5} & \frac{-9}{5} \end{bmatrix} \left(R_2 \to R_2 - \frac{1}{5} R_1 \right)$$

$\because$ $\begin{vmatrix} 5 & 2 \\ 0 & \frac{-7}{5} \end{vmatrix} \neq 0$ is a minor of order 2 of A and

$\begin{vmatrix} 5 & -1 \\ 0 & \frac{-9}{5} \end{vmatrix} \neq 0$ is a minor of order 2 of (A, B) .

$\rho(A) = \rho(A, B) = 2 <$ number of unknowns,

$\therefore$ The system is consistent and has infinite solutions which involve $4 - 2 = 2$ arbitrary constants.

The final - reduced matrix makes the system as,

$$5x + 2y - 6z + 2t = -1 \quad .....(1.35)$$

and $$\frac{-7}{5}y + \frac{11}{5}z \frac{-7}{5}t = \frac{-9}{5}$$

(or) $$-7y + 11z - 7t = -9 \quad .....(1.36)$$

Taking $z = k_1$, $t = k_2$, (k_1, k_2 being arbitrary constants)

we get, from (1.36),

$$y = \frac{1}{7}(11k_1 - 7k_2 + 9)$$

$$(1.35) \Rightarrow \quad x = \frac{1}{5}(-2y + 6z - 2t - 1)$$

$$= \frac{1}{5}\left[\frac{-2}{7}(11k_1 - 7k_2 + 9) + 6k_1 - 2k_2 - 1\right]$$

$$= \frac{1}{7}(4k_1 - 5).$$

$\therefore$ The totality of solutions of the system is

$$x = \frac{1}{7}(4k_1 - 5),\ y = \frac{1}{7}(11k_1 - 7k_2 + 9),\ (z = k_1,\ t = k_2$$

where k_1, k_2 are arbitrary constants).

***Example* 1.30:** Solve the system

$$x + y + z = 6$$
$$2x - 3y + 4z = 8$$
$$x - y + 2z = 5, \text{ by the methods}$$

(i) Matrix inversion method
(ii) Using cramer's rule
(iii) Gauss Jordan method

***Solution*:** The system is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 5 \end{bmatrix}$$

If it is put in the form AX = B, we have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 1 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 8 \\ 5 \end{bmatrix}$$

(i) Matrix inversion method :

We find A^{-1}

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

Adjoint of $$A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

where A_{11}, A_{12}, ... are cofactors of a_{11}, a_{12}, ... respectively.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -3 & 4 \\ -1 & 2 \end{vmatrix} = -2 \; ; A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 0;$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} = +1;$$

$$|A| = \det A = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$
$$= 1.(-2) + 1(0) + 1\,(1) = -1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 1 \; ; A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2$$

$$A_{31} = (-1)^{1+3}\begin{vmatrix} 1 & 1 \\ -3 & 4 \end{vmatrix} = 7 \; ; A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -2 \; ;$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$$

$$\therefore \quad \text{Adj}\, A = \begin{bmatrix} -2 & -3 & 7 \\ 0 & 1 & -2 \\ -1 & 2 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A = -\begin{bmatrix} -2 & -3 & 7 \\ 0 & 1 & -2 \\ 1 & 2 & -5 \end{bmatrix}$$

Solution matrix

$$X = A^{-1} B$$

$$\therefore \quad X = \frac{+1}{-1}\begin{bmatrix} -2 & -3 & 7 \\ 0 & 1 & -2 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 5 \end{bmatrix}$$

$$= -\begin{pmatrix} -2.6 - 3.8 + 7.5 \\ 0.6 + 1.8 - 2.5 \\ 1.6 + 2.8 - 5.5 \end{pmatrix}$$

$$= -\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ x = 1, y = 2, z = 3 is the solution.

(ii) Cramer's Rule :

$$\Delta = |A| = -1$$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 8 & -3 & 4 \\ 5 & -1 & 2 \end{vmatrix} = 6(-6 + 4) - 1(16 - 20) + 1(-8 + 15)$$

$= -12 + 4 + 7 = -1$ Replacing 1st column of Δ by B

Similarly, $$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 8 & 4 \\ 1 & 5 & 2 \end{vmatrix} = 1(16 - 20) - 6(4 - 4) + 1(10 - 8)$$

$$= -4 - 0 + 2 = -2$$

and $$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 2 & -3 & 8 \\ 1 & -1 & 5 \end{vmatrix} = 1(-15 + 8) - 1(10 - 8) + 6(-2 + 3)$$

$$= -7 - 2 + 6 = -3$$

$\therefore$ solution is

$$x = \frac{\Delta_1}{\Delta} = 1; \; y = \frac{\Delta_2}{\Delta} = 2; \; z = \frac{\Delta_3}{\Delta} = 3$$

(iii) Gauss-Jordan method :

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & -3 & 4 & 8 \\ 1 & -1 & 2 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -5 & 2 & -4 \\ 0 & -2 & 1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -2/5 & \frac{4}{5} \\ 0 & -2 & 1 & -1 \end{bmatrix} \quad R_2 \to -\frac{1}{5} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 7/5 & 26/5 \\ 0 & 1 & -2/5 & 4/5 \\ 0 & 0 & 1/5 & 3/5 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 7/5 & 26/5 \\ 0 & 1 & -2/5 & 4/5 \\ 0 & 0 & 1 & 3 \end{bmatrix} R_3 \to 5R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{7}{5} R_3 \\ \\ R_2 \to R_2 + \frac{2}{5} R_3 \end{matrix}$$

$$= [I_3 \; X]$$

where $$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ solution is

$$x = 1, y = 2, z = 3$$

***Example* 1.31:** Show that the system

$$2x - 3y + z = 0,$$
$$4x + 9y + z = 0,$$
$$8x - 27y + z = 0,$$

has no non-trivial solution.

***Solution*:** The given system can be written as

$$\begin{bmatrix} 2 & -3 & 1 \\ 4 & 9 & 1 \\ 8 & -27 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which is of the form

$$AX = 0,$$

where $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 9 & 1 \\ 8 & -27 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 9 & 1 \\ 8 & -27 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3/2 & 1/2 \\ 4 & 9 & 1 \\ 8 & -27 & 1 \end{bmatrix} R_1 \to \frac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 15 & -1 \\ 0 & -15 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 8R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & -1/15 \\ 0 & -15 & -3 \end{bmatrix} R_2 \to \frac{1}{15}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & -1/15 \\ 0 & 0 & -4 \end{bmatrix} \begin{matrix} R_1 \to R_1 + \frac{3}{2}R_2 \\ R_3 \to R_3 + 15R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & -1/15 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to \frac{-1}{4}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2/5 R_3 \\ R_2 \to R_2 + 1/15 R_3 \end{matrix}$$

which is in the canonical form I_3.

$\therefore$ $\rho(A) = 3$ = number of unknowns.

$\therefore$ The system is consistent and has unique solution which is the trivial solution $x = 0,\ y = 0,\ z = 0$.

(ALITER : Since A is a 3×3 matrix, it is easy to find $\rho(A)$ by showing that $|A| \neq 0$.)

***Example* 1.32:** Solve the system of homogeneous equations given by

$$2x + y + 2z = 0,$$
$$x + y + 3z = 0,$$
$$4x + 3y + 8z = 0.$$

Solution: The given system is of the form AX = 0 where

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & 8 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 8 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix} \begin{matrix} R_2 \to -R_2 \\ R_3 \to -R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2,$$

$\therefore$ $\rho(A) = 2$,

Hence the system is consistent and has infinite solutions with one arbitrary constant.

The final reduced matrix reduces the system to

$$x + y + 3z = 0$$
$$y + 4z = 0$$

Taking z = k, we get y = – 4k, and x = k (k being arbitrary constant)

∴ The solution of the system is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -4k \\ k \end{bmatrix}$$

***Example* 1.33:** Solve the system,

$$x + y = 0,$$
$$x + 2y + 3z = 0,$$
$$u + v + w = 0,$$
$$x + u + w = 0.$$

Solution: In this problem, we have to find 6 variables x, y, z, u, v, w.

The given system is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If written in the form AX = 0, we have,

$$\text{we find that,} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0, \text{ is the largest non- vanishing minor of A.}$$

∴ $\rho(A) = 4$

Hence the system has infinite solutions and involves 6 – 4 = 2 arbitrary constants.

Taking $v = k_1, w = k_2$ (k_1, k_2 being arbitrary constants)

We have, $u = -k_1 - k_2$

$x = k_1, y = -k_1$

$$z = \frac{1}{3} k_1$$

∴ The solution is

$x = +k_1,$

$y = -k_1,$

$$z = \frac{1}{3} k_1,$$

$u = -k_1 - k_2,$

$v = k_1,$

$w = k_2$

Note : ρ(A) can also be found by any alternate method.

***Example* 1.34:** If a, b, c, are distinct non-zero numbers, show that the homogeneous system

$$\begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has no non-trivial solution.

Solution: Here AX = 0 is the system where

$$A = \begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} \text{ w}$$

$$= abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \begin{matrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{matrix}$$

$$= abc\,.\,(b-a)\,(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix} \begin{matrix} c_2 \to \frac{1}{b-a}c_2 \\ c_3 \to \frac{1}{c-a}c_3 \end{matrix}$$

$$= abc\,(b-a)\,(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$$

$= abc\,(b-a)\,(c-a)\,(c-b) \neq 0$, since a, b, c are distinct non-zero numbers.

$\therefore$ $\rho(A) = 3$ = number of unknowns. Hence the system has a unique trivial solution $x = 0, y = 0, z = 0$.

Hence the problem ($\rho(A)$ can also be found by any other method).

Exercise - 1.4

I Test for consistency the following systems. If consistent, solve them.

(i) $x - 2y + t = 4$; $3x + 5y + t = 6$, $6x - y + 4t = 2$

(Ans : not consistent**)**

(ii) $4x - y + 3z = 11$, $2x + y - 3z = -5$, $x + y + z = 6$

(Ans : $x = 1, y = 2, z = 3$**)**

(iii) $x - y + z = 3$, $2x - 3y + 5z = 10$, $x + y + 4z = 4$

(Ans : $x = 1, y = -1, z = 1$**)**

(iv) $x_1 + x_2 - x_3 + x_4 = 2$; $3x_1 - x_2 + 2x_3 + 5x_4 = 9$

$4x_1 + x_2 - x_3 - x_4 = 3$; $2x_1 - x_2 + 3x_3 + x_4 = 5$

(Ans : $x_1 = x_2 = x_3 = x_4 = 1$**)**

1.7 Eigen Values and Eigen Vectors

Matrix Polynomial

Let P_0, P_1, P_2 ..... P_n ($P_n \neq 0$) be square matrices of same order, then an expression of the form

$$P_0 + P_1x + P_2x^2 + \ldots\ldots + P_nx^n,$$

is called a matrix polynomial of order n.

***Example* 1**

Let $P_0 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, P_1 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix},$

Then,
$$P_0 + P_1x + P_2x^2 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}x + \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}x^2$$
$$= \begin{bmatrix} (1+x+2x^2) & (2+2x+3x^2) \\ (3+3x+x^2) & (x+4x^2) \end{bmatrix}$$

is a matrix polynomial of order 2.

Characteristic matrix

Let (1) A be a square matrix of order 'n' (2) I be a unit matrix of order n. Then the matrix polynomial given by $(A - \lambda I)$ is called the characteristic matrix of A.

***Example* 2**

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then the polynomial

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

is the characteristic matrix of A.

Characteristic equation of A

For a square matrix A the equation $|A - \lambda I| = 0$ is called the characteristic equation.

***Example* 3**

In the example 2, the characteristic equation is given by

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda - 3 = 0 \cdot$$

Eigen Values

The roots of the characteristic equation are called the characteristic values or roots or Eigen values or Latent roots of the square matrix.

In example 3, the eigen values of $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ are the roots of the equation $\lambda^2 - 2\lambda - 3 = 0$

i.e., $\lambda = 3$ and -1 are the eigen values.

Eigen vectors

Let λ be an eigen value of the square matrix A.

Then, a non-zero column vector X = $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that

$(A - \lambda I)X = 0_{n\times 1}$

(or) $AX = \lambda IX$

(or) $AX = \lambda X$, is called an eigen vector of A corresponding to λ.

Properties of Eigen Values and Eigen Vectors

1. The sum of the eigen values of a square matrix A is its trace and their product is $|A|$.

[JNTU 2003s, 2004s]

***Proof* :**

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n and I be the unit matrix of same order.

The eigen values λ_i, $i = 1, 2, \ldots n$, are the roots of the equation

$$|A - \lambda I| = \begin{vmatrix} (a_{11} - \lambda) & a_{12} & \cdots & a_{1n} \\ a_{21} & (a_{22} - \lambda) & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & (a_{nn} - \lambda) \end{vmatrix} = 0 \qquad \ldots\ldots(1.37)$$

Expanding the determinant let

$$|A - \lambda I| = p_0\lambda^n + p_1\lambda^{n-1} + ... + p_n \qquad .....(1.38)$$

where $p_0, p_1, p_2, ...p_n$ are constants.

Taking $\lambda = 0$ in (1.38), we get, $|A| = p_n$ .....(1.39)

Expanding $|A - \lambda I|$ by first row, we get from (1.37),

$$|A - \lambda I| = (-1)^n \lambda^n + (-1)^{n-1} \lambda^{n-1}(a_{11} + a_{22} + ... + a_{nn}) + + ... \qquad .....(1.40)$$

Comparing (1.38) and (1.40), we have,

$$p_0 = (-1)^n, \; p_1 = (-1)^{n-1}(a_{11} + a_{22} + ... + a_{nn}), \; ....$$

Sum of eigen values $= \lambda_1 + \lambda_2 + ... + \lambda_n$

$$= \frac{-p_1}{p_0}$$

$$= \frac{(-1)^n(a_{11} + a_{22} + ... + a_{nn})}{(-1)^n}$$

$$= a_{11} + a_{22} + ... + a_{nn}$$

= Trace of A.

Product of eigen values $= \lambda_1 \lambda_2 ..... \lambda_n$

$$= (-1)^n . \frac{p_n}{p_0}$$

$$= \frac{(-1)^n p_n}{(-1)^n} = p_n = |A| \qquad \text{from (1.39).}$$

Note 1 :

$\because \qquad |A| = \lambda_1\lambda_2 ... \lambda_n$,

(i) If A is non-singular,

$|A| \neq 0$

$\Rightarrow \lambda_1\lambda_2 ...\lambda_n \neq 0 \quad \Rightarrow$ all eigen values are non-zero numbers.

(ii) If A is singular, $|A| = 0 = \lambda_1 \lambda_2 ... \lambda_n \Rightarrow$ at least one eigen value of A must be equal to zero.

2. The eigen values of A and A^T are equal.

Proof :

For any square matrix B, $|B| = |B^T|$.

$\therefore \quad |A - \lambda I| = |(A - \lambda I)^T| = |A^T - (\lambda I)^T| = |A^T - \lambda I|$ which proves the result.

3. If A is a non-singular matrix and λ is an eigen value of A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}. Hence deduce that $\frac{|A|}{\lambda}$ is an eigen value of adj A **[JNTU Nov 2005]**

Proof :

Let X be the eigen vector corresponding to λ.

Then, $AX = \lambda X$

$\therefore \quad A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow \quad IX = \lambda A^{-1}X$

$\Rightarrow \quad A^{-1}X = \left(\frac{1}{\lambda}\right)X$

$\Rightarrow \quad \frac{1}{\lambda}$ is an eigen value of A^{-1}.

Therefore we have

$$|A| A^{-1} X = \frac{|A|}{\lambda} X$$

$$\Rightarrow \quad (\text{Adj } A) X = \frac{|A|}{\lambda} X \qquad .....(1.41)$$

$\Rightarrow \quad \frac{|A|}{\lambda}$ is eigen value of (AdjA)

Corollary : If λ_i, $i = 1, 2, \ldots, n$ are eigen values of A ($|A| \neq 0$), then $\frac{1}{\lambda_i}$, $i = 1, 2, \ldots, n$ are eigen values of A^{-1}.

4. If λ is an eigen value of A, then $\mu\lambda$ is an eigen value of μA where μ is a non-zero scalar

Proof :

X be an eigen vector of A corresponding to λ.

Then $$AX = \lambda X$$
$$(\mu A)X = \mu\,(AX) = \mu\,(\lambda X) = (\mu\lambda)X$$

which prove the result

5. If λ is an eigen value of A, then λ^m is an eigen value of A^m, m being any positive integer

[JNTU 2000, 2003]

Proof :

Let X be an eigen vector of A corresponding to λ.

Then $$AX = \lambda X \qquad .....(1.42)$$
$$A^2X = A(AX) = A(\lambda X) = \lambda(AX)$$
$$= \lambda(\lambda X) \qquad \text{from } (1.42)$$
$$= \lambda^2 X$$

$\therefore$ λ^2 is an eigen value of A^2. Proceeding the same way, the result can be proved.

Corollary : If λ_i, $i = 1, 2, ... n$ are eigen values of A, λ_i^m, $i = 1, 2 ..... n$, are eigen values of A^m where m is any positive integer.

6. The eigen values of a diagonal matrix are its diagonal elements

Proof :

Let A = Diagonal $(a_{11}, a_{22} ..... a_{nn})$.

Then $$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & 0 & 0 & \cdots & 0 \\ 0 & a_{22}-\lambda & 0 & \cdots & 0 \\ 0 & 0 & a_{33}-\lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (a_{11}-\lambda)\,(a_{22}-\lambda)\cdots(a_{nn}-\lambda) = 0$$

$\Rightarrow$ $\lambda = a_{11}, a_{22}, \cdots a_{nn}$; which completes the proof.

7. If B be a non-singular matrix, and A, B are matrices of same order, then A and $B^{-1}AB$ have same eigen values

Proof :

A and B are of same order.

Let $\qquad C = B^{-1} AB$

Then $\qquad C - \lambda I = B^{-1}AB - \lambda I = B^{-1}AB - \lambda B^{-1}B$

$$= B^{-1}AB - B^{-1}\lambda B$$

$$= B^{-1}(AB - \lambda IB)$$

$$= B^{-1}(A - \lambda I)B$$

$$|C - \lambda I| = |B^{-1}||A - \lambda I||B|$$

$$|C - \lambda I| = |A - \lambda I| \qquad \left(\because |B^{-1}||B| = |B^{-1}B| = |I| = 1\right)$$

$\therefore$ $\quad A$ and $C = B^{-1}AB$ have same eigen values.

8. λ is a characteristic root of a square matrix A if and only if there exists a non-zero vector X such that $AX = \lambda X$.

Proof :

Let λ be a characteristic root of A.

$\therefore \qquad |A - \lambda I| = 0$

$\Rightarrow \qquad (A - \lambda I)$ is a singular matrix.

$\therefore$ $\quad$ The homogeneous system of equations given by $(A - \lambda I)X = 0$ possesses non-zero solutions.

i.e., $\quad$ there exists a non-zero vector X such that

$$(A - \lambda I)X = 0$$

$\Rightarrow \qquad AX = \lambda IX = \lambda X$

conversely, $\qquad AX = \lambda X$

$\Rightarrow \qquad (A - \lambda I)X = 0$

where X is a non-zero vector.

$\therefore$ $\quad$ The system of homogeneous equations $(A - \lambda I)X = 0$ has a non-zero solution.

Hence the coefficient matrix $(A - \lambda I)$ is singular.

i.e., $|A - \lambda I| = 0$

which shows that λ is an eigen value of A.

9. If X be an eigen vector of A corresponding to the eigen value λ, then cX is also an eigen vector of A corresponding to λ, c being a non-zero scalar

Proof :

X is an eigen vector of A corresponding to λ.

$\therefore \quad AX = \lambda X$

$\Rightarrow \quad cAX = c\lambda X$

$\Rightarrow \quad A(cX) = \lambda(cX)$

which proves the result.

10. If X is an eigen vector of a square matrix A, then X cannot correspond to more than one eigen value of A.

Proof :

If possible, let X correspond to two eigen values λ_1 and λ_2 of A.

Then, we have,

$$AX = \lambda_1 X \quad \text{and} \quad AX = \lambda_2 X$$

$\therefore \quad \lambda_1 X = \lambda_2 X \Rightarrow (\lambda_1 - \lambda_2)X = 0$

$\Rightarrow \quad \lambda_1 = \lambda_2 \quad (\because X \neq 0)$

Hence the result.

11. Zero is an eigen value of a matrix if and only if it is singular

Proof :

Let $\lambda = 0$ be an eigen value of a matrix 'A'.

Then it satisfies the equation

$$|A - \lambda I| = 0 \qquad .....(1.43)$$

$\Rightarrow \quad |A| = 0 \quad \Rightarrow \quad A$ is singular.

Conversely, A is singular

$\Rightarrow \quad |A| = 0$

$\Rightarrow \quad \lambda = 0$ satisfies the equation (1.43)

$\Rightarrow \quad \lambda = 0$ is an eigen value of A.

12. If λ is an eigen value of a non-singular matrix A, show that $\frac{|A|}{\lambda}$ is an eigen value of $Adj\ A$.

Proof :

A is non-singular $\Rightarrow \lambda \neq 0$.

$X \neq 0$ be the eigen vector corresponding to λ.

Then $\quad AX = \lambda X$

$\Rightarrow \quad (Adj\ A)\ AX = (Adj\ A)\lambda X$

$\Rightarrow \quad |A|\,X = (Adj\ A)\lambda X \qquad [\because (Adj\ A)A = |A|\,I]$

$\Rightarrow \quad \frac{|A|}{\lambda} X = (Adj\ A)X$

(or) $\quad (Adj\ A)X = \frac{|A|}{\lambda} X$ which proves the result.

Solved Examples

***Example* 1.35:** If $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 7 & 0 \\ 2 & 6 & 1 \end{bmatrix}$, find the eigen values of (i) A^{-1} (ii) A^3 and (iii) A^T.

Solution: Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 3-\lambda & 0 & 0 \\ 5 & 7-\lambda & 0 \\ 2 & 6 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (3-\lambda)\ (7-\lambda)\ (1-\lambda) = 0$$

$\Rightarrow \quad \lambda = 3, 7, 1$ are eigen values of A.

$\therefore$ (i) Eigen values of A^{-1} are $\frac{1}{1}, \frac{1}{3}, \frac{1}{7},$

i.e., $1, \frac{1}{3}, \frac{1}{7}.$

(ii) Eigen values of A^3 are $1^3, 3^3, 7^3$.

(iii) Eigen values of A^T are same as those of A,

i.e., 1, 3, 7.

***Example* 1.36:** Without finding the actual eigen values of $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, find their sum and product.

Solution: $|A| = 15,$

sum of eigen vectors of A = Trace of A

$= 1 + 3 + 5 = 9$

Product = $|A| = 15$

***Example* 1.37:** Find the characteristic values of the matrices,

(i) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ and (ii) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{pmatrix}$.

Solution: The given matrix be A. Then

(i) Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad (\cos\alpha - \lambda)^2 - \sin^2\alpha = 0$$

$$\Rightarrow \quad (cos\alpha - \lambda + sin\alpha)\ (cos\alpha - \lambda - sin\alpha) = 0$$

$\therefore$ $\lambda = \cos\alpha + \sin\alpha, \cos\alpha - \sin\alpha$ are eigen values.

(ii)

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} \cos\alpha - \lambda & -\sin\alpha \\ -\sin\alpha & -\cos\alpha - \lambda \end{vmatrix} = 0$$

$$-(\cos^2\alpha - \lambda^2) - \sin^2\alpha = 0$$

$$\Rightarrow \quad \lambda^2 = \cos^2\alpha + \sin^2\alpha = 1$$

$\lambda = \pm 1$ are eigen values.

***Example* 1.38:** Find the eigen values of the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 4 \end{bmatrix}$.

Solution: $|A - \lambda I| = \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 4-\lambda \end{vmatrix}$

$$= (-9-\lambda)\{\lambda^2 - 7\lambda - 20\} - 4(-32 + 8\lambda + 64) + 4(-64 + 48 - 16\lambda)$$

$$= -9\lambda^2 + 63\lambda + 180 - \lambda^3 + 7\lambda^2 + 20\lambda - 32\lambda - 128 - 64\lambda - 64$$

$$= -\lambda^3 - 2\lambda^2 - 13\lambda - 12$$

Characteristic equation of A is

$|A - \lambda I| = 0$

$\Rightarrow \quad \lambda^3 + 2\lambda^2 + 13\lambda + 12 = 0$

By inspection, $\lambda = -1$ is a root of this equation.

The above equation can be written as

$(\lambda + 1)\ (\lambda^2 + \lambda + 12) = 0$

$\lambda^2 + \lambda + 12 = 0$

$\Rightarrow \lambda = \dfrac{-1 \pm \sqrt{1-48}}{2}$

$= \dfrac{-1 \pm \sqrt{47}i}{2}$

$\therefore$ Eigen values of A are -1, $\dfrac{-1 \pm \sqrt{47}i}{2}$

***Example* 1.39:** Find the eigen values of the matrix $A = \begin{bmatrix} 3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: $A - \lambda I = \begin{bmatrix} 3-\lambda & -9 & -12 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

$$|A-\lambda I| = (1-\lambda)\left[(3-\lambda)^2+9\right] \qquad \text{(expanding by last row)}$$

$$= (1-\lambda)\left(\lambda^2-6\lambda+18\right)$$

Characteristic equation of A is

$$|A-\lambda I| = 0$$

$\Rightarrow$ $$(1-\lambda)(\lambda^2-6\lambda+18) = 0$$

$\therefore$ Eigen values of A are

$$\lambda = 1, \ \frac{6 \pm \sqrt{36-72}}{2}$$

i.e., $\lambda = 1,\ 3+3i,\ 3-3i$

***Example* 1.40:** Find the eigen values and corresponding eigen vectors of the unit matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: If A is a square matrix, its eigen values 'λ' are given by the characteristic equation $|A-\lambda I| = 0$.

$\therefore$ In this problem, the characteristic equation is $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0.$

$\Rightarrow$ $(1-\lambda)^3 = 0$,

$\therefore$ Eigen values are $\lambda = 1, 1, 1$

Eigen vectors are given by,

$AX = \lambda X$

$\Rightarrow$ $IX = X$ .....(1.44)

$\because$ (1.44) is satisfied by all non-zero vectors X, all non-zero vectors are the eigen vectors of the given unit matrix.

***Example* 1.41:** Determine the eigen values and corresponding eigen vectors of A = $\begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$.

Solution: $$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ 1 & 5-\lambda \end{vmatrix} = \lambda^2-7\lambda+6$$

Characteristic equation is $|A - \lambda I| = 0$

i.e., $\lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow$ $\lambda = 1, 6$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector.

***Case* (i):** $\lambda = 1$

$AX = \lambda X$

$\Rightarrow$ $AX = X$

$$\Rightarrow \quad \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow$ $x_1 + 4x_2 = 0$; $x_1 + 4x_2 = 0$

$\therefore$ $x_2 = k$

$\Rightarrow$ $x_1 = -4k$

$\therefore$ $X = \begin{bmatrix} -4k \\ k \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

***Case* (ii) :** $\lambda = 6$

$\Rightarrow$ $AX = 6X$

$$\Rightarrow \quad \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 \\ 6x_2 \end{bmatrix}$$

$-4x_1 + 4x_2 = 0$

$x_1 - x_2 = 0$

Both these equations are equivalent to one equation

$x_1 - x_2 = 0$

$$\Rightarrow \qquad x_1 = x_2 = c$$

(say) where c is a constant.

$$\therefore \qquad X = \begin{bmatrix} c \\ c \end{bmatrix}$$ is the eigen vector corresponding to $\lambda = 6$.

***Example* 1.42:** Find the eigen values and corresponding eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution: Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \qquad \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

which on expansion and simplification gives

$$\lambda^3 - 7\lambda^2 + 36 = 0$$

$$\Rightarrow \qquad (\lambda+2)\ (\lambda-6)\ (\lambda-3) = 0$$

$$\Rightarrow \qquad \lambda = -2,\ 3,\ 6 \text{ are eigen values of } A.$$

***Case (i)* :**

To find the eigen vector when $\lambda = -2$:

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector, so that $AX = -2X$

$$\Rightarrow \qquad \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \end{bmatrix}$$

$$\Rightarrow \qquad 3x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 7x_2 + x_3 = 0$$

$$3x_1 + x_2 + 3x_3 = 0,$$

which are equivalent to only two equations. Solving the first two equations by rule of cross-multiplication, we have

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} = c \text{ (say)}$$

$\Rightarrow \quad x_1 = -c, x_2 = 0, x_3 = c \quad$ (c being arbitrary constant)

$\Rightarrow \quad x = \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix}$ is eigen vector for $\lambda = -2$. If $c = 1$ is a particular value,

$x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the corresponding eigen vector.

Case (ii) :

When $\lambda = 3$; $AX = 3X$

i.e.,
$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} -2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ 3x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

The first two equations give,

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} = \text{c (say)}$$

$\Rightarrow \quad x_1 = c, x_2 = -c, x_3 = $ c, (which satisfy the third equation also).

$\therefore \quad X = \begin{bmatrix} c \\ -c \\ c \end{bmatrix}$ is the eigen vector.

Taking $c = 1, X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is eigen vector for $\lambda = 3$.

Case (iii) :

When $\lambda = 6$; $AX = 6X$

$$\Rightarrow \quad \begin{aligned} -5x_1 + x_2 + 3x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 - 5x_3 &= 0 \end{aligned}$$

Solving the first two equations,

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} = \text{c (say)}$$

$\Rightarrow$ $x_1 = c, x_2 = 2c, x_3 = c$ (these satisfy third equation)

$\therefore$ $X = \begin{bmatrix} c \\ 2c \\ c \end{bmatrix}$ is the eigen vector and a particular vector is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

***Example* 1.43:** Find the eigen values and corresponding eigen vectors of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

Solution:

$$|A - \lambda I| = \begin{vmatrix} (1-\lambda) & 2 & 3 \\ 0 & (-4-\lambda) & 2 \\ 0 & 0 & 7-\lambda \end{vmatrix} = (1 - \lambda)(-4 - \lambda)(7 - \lambda)$$

Characteristic equation of A is

$$(1 - \lambda)(-4 - \lambda)(7 - \lambda) = 0$$

$\therefore$ Eigen values are $\lambda = 1, -4, 7$.

To find eigen vectors.

Case (i) : When $\lambda = 1$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to $\lambda = 1$.

$$\therefore \quad AX = X \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} 2x_2 + 3x_3 &= 0 \\ -5x_2 + 2x_3 &= 0 \\ 6x_3 &= 0 \end{aligned}\right\} \qquad \text{.....(1.45)}$$

The solution of the system (1.45) is

$$x_3 = 0, x_2 = 0, x_1 = c$$

$$\therefore \quad \text{Eigen vector} = X = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}, \text{ (c being constant)}$$

In particular $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Case (ii) : Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to $\lambda = -4$.

$$\therefore \quad AX = -4X$$

$$\Rightarrow \quad 5x_1 + 2x_2 + 3x_3 = 0,\ 2x_3 = 0,\ 11x_3 = 0$$

$$\Rightarrow \quad x_3 = 0,\ x_2 = k,\ x_1 = \frac{-2k}{5}$$

A particular solution is $x_1 = \frac{-2}{5}$, $x_2 = 1$, $x_3 = 0$

Case (iii) : Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to $\lambda = 7$.

$$\therefore \quad AX = 7X$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7x_1 \\ 7x_2 \\ 7x_3 \end{bmatrix}$$

$$\Rightarrow \qquad \left.\begin{aligned} -6x_1 + 2x_2 + 3x_3 &= 0 \\ -11x_2 + 2x_3 &= 0 \end{aligned}\right\} \qquad .....(1.46)$$

Taking $x_3 = c$, c being constant, the solution of (1.46) is

$$x_3 = c,\ x_2 = \frac{2}{11}\ c,\ x_1 = \frac{1}{6}\left[\frac{4c}{11} + 3c\right]$$

$$\therefore \qquad X = \begin{bmatrix} \frac{1}{6}\left(\frac{4c}{11} + 3c\right) \\ \frac{2c}{11} \\ c \end{bmatrix} = \begin{bmatrix} \frac{37c}{66} \\ \frac{2c}{11} \\ c \end{bmatrix} \text{ is the eigen vector for } \lambda = 7.$$

***Example* 1.44:** Determine the characteristic roots and vectors of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

[CSVTU 2005, 2007, 2008]

Solution:

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}$$

$$= (8 - \lambda)[\lambda^2 - 10\lambda + 5] + 6\ [6\lambda - 10] + 2[10 + 2\lambda]$$

$$= -\lambda^3 + 18\lambda^2 - 45\lambda$$

Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \qquad -\lambda^3 + 18\lambda^2 + 45\lambda = 0$$

$$\Rightarrow \qquad \lambda(\lambda - 3)(\lambda - 15) = 0$$

$\therefore$ Characteristic values are $\lambda = 0,\ 3,\ 15$

Let us find the corresponding characteristic vectors

Let $\qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the required vector.

(i) when $\qquad \lambda = 0,\ \therefore\ AX = \lambda X = 0$

$$\Rightarrow \qquad \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad \left.\begin{array}{rcrcrcl} 8x_1 & - & 6x_2 & + & 2x_3 & = & 0 \\ -6x_1 & + & 7x_2 & - & 4x_3 & = & 0 \\ 2x_1 & - & 4x_2 & + & 3x_3 & = & 0 \end{array}\right\}$$

Solving the first two equations, we have,

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\Rightarrow \quad x_1 = \frac{x_2}{2} = \frac{x_3}{2} = c \text{ (say)}$$

$$\Rightarrow \quad x_1 = c\ ;\ x_2 = 2c,\ x_3 = 2c$$

(these values satisfy the third equation also)

$$\therefore \quad X = \begin{bmatrix} c \\ 2c \\ 2c \end{bmatrix}$$

where c is a constant.

A particular value $= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

(ii) when $\lambda = 3\ ;\ AX = 3X$

$$\Rightarrow \quad \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{array}{rcrcrcl} 5x_1 & - & 6x_2 & + & 2x_3 & = & 0 \\ -6x_1 & + & 4x_2 & - & 4x_3 & = & 0 \\ & & 2x_1 & - & 4x_2 & = & 0 \end{array}$$

Solving first two equations, we get

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} \Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = c \text{ (say)}$$

$\therefore \quad x_1 = 2c,\ x_2 = c,\ x_3 = -2c$ and these values satisfy the 3rd equation.

$\therefore \qquad X = \begin{bmatrix} 2c \\ c \\ -2c \end{bmatrix}$ is the required vector where c is a constant.

A particular vector can be $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

(iii) When $\lambda = 15$; $AX = 15X$

i.e.,
$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15x_1 \\ 15x_2 \\ 15x_3 \end{bmatrix}$$

By multiplication and equality of matrices, we have

$$-7x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 - 8x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 - 12x_3 = 0$$

From the first two equations, we have,

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = c \text{ (say)}$$

$\therefore \quad x_1 = 2c,\ x_2 = -2c,\ x_3 = c,$ where c is a constant.
These values satisfy the 3rd equation also

$\therefore \quad X = \begin{bmatrix} 2c \\ -2c \\ c \end{bmatrix}$ is the required vector and a particular value can be taken as $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

***Example* 1.45:** Find the latent roots and the corresponding latent vectors of the matrix

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}.$$

Solution:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}$$

$$= (2 - \lambda)\ \{-(1+\lambda)\ (1-\lambda)-3\}$$
$$+\ 2(-1-\lambda-1)\ +\ 2(3-1+\lambda)$$
$$= (2-\lambda)\ (\lambda^2-4)-2(2+\lambda)+2(2+\lambda)$$
$$= -(\lambda-2)\ (\lambda^2-4)$$

Characteristic equation is $|A - \lambda I| = 0$

i.e., $(\lambda - 2)\ (\lambda - 2)\ (\lambda + 2) = 0$

$\Rightarrow$ Latent roots of A are $\lambda = 2, 2, -2$

Case (i) : $\lambda = 2$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the corresponding latent vector.

$\therefore$ $AX = 2X$

$$\Rightarrow \quad \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

Applying matrix multiplication and equality of matrices, we get on simplification,

$$-x_2 + x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$x_1 + 3x_2 - 3x_3 = 0$$

From the first two equations

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1} = c \text{ (say)}$$

$\therefore$ $x_1 = 0, x_2 = c, x_3 = c$ which satisfy the 3rd equation also

$X = \begin{bmatrix} 0 \\ c \\ c \end{bmatrix}$ is the required vector.

case (ii) when $\lambda = -2$, we have $AX = -2X$

$$\Rightarrow \quad \begin{aligned} 4x_1 - 2x_2 + 2x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \end{aligned}$$

$$\Rightarrow \quad \frac{x_1}{-8} = \frac{x_2}{-2} = \frac{x_3}{14}$$

$$\Rightarrow \quad \frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7} = c$$

Required vector is $\begin{bmatrix} -4c \\ -c \\ 7c \end{bmatrix}$ or in particular $\begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix}$.

***Example* 1.46:** Determine the constants p, q, r, s, t, u so that [1 1 1]', [1 0 –1]' and [1 –1 0]' are the eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix}$, let l_1, l_2 l_3 be the eigen values of A.

$[1\ 1\ 1]' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the eigen vector corresponding to λ_1.

$$\therefore \quad AX_1 = \lambda_1 X$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_1 \end{bmatrix}$$

$1.1 + 1.1 + 1.1 = \lambda_1$; i.e., $\lambda_1 = 3$

$\Rightarrow$ $p.1 + q.1 + r.1 = \lambda_1 \Rightarrow \quad p + q + r = 3$ .....(1.47)

$s.1 + t.1 + u.1 = \lambda_1 = \lambda_1 \quad s + t + u = 3$ .....(1.48)

Let $X = [1\ 0\ -1]' = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ be the eigen vector corresponding to λ_2.

Then $AX = \lambda_2 X$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ 0 \\ -\lambda_2 \end{bmatrix}$$

$\Rightarrow \quad 1.1 + 1.0 + 1.-1 = \lambda_2 \quad$ (i.e.,) $\lambda_2 = 0$

$p.1 + q.0 + r.-1 = 0 \quad \Rightarrow \quad p - r = 0 \qquad$ .....(1.49)

$s.1 + t.0 + u.-1 = -\lambda_2 = 0 \Rightarrow \quad s - u = 0 \qquad$ .....(1.50)

Similarly,
$$\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \lambda_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_3 \\ -\lambda_3 \\ 0 \end{bmatrix}$$

gives, $\lambda_3 = 0$,

$p - q = 0 \qquad$ .....(1.51)

and $\quad s - t = 0 \qquad$ .....(1.52)

To get the values of p, q, r, s, t, u we have to solve the equations (1.47) to (1.52).

(1.47) + (1.49) + (1.51) $\Rightarrow$ 3p = 3 $\Rightarrow$ p = 1

$\therefore \quad$ (1.49) $\Rightarrow$ r = 1, and (1.51) $\Rightarrow$ q = 1.

Similarly, from (1.48), (1.50) and (1.52) we get s = t = u = 1.

$\therefore \quad p = q = r = s = t = u = 1.$

***Example* 1.47:** If A and B are non – singular matrices of same order , show that AB and BA have same eigen values.

Solution: $AB = (B^{-1}B)(AB)$ [since $B^{-1}B = I$] $= B^{-1}(BA)B$ have same eigen values. i.e., BA and AB have same eigen values.

***Example* 1.48:** Show that the eigen values of a triangular matrix are just the diagonal elements of the matrix.

Solution: Let $A=\begin{bmatrix} a_{11} & a_{12}......a_{1m} \\ 0 & a_{22}.......a_{2m} \\ .... \\ .... \\ 0......0.......a_{mm} \end{bmatrix}$ be triangular matrix.

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} (a_{11}-\lambda) & a_{12}...........a_{1m} \\ 0 & (a_{22}-\lambda).......a_{2m} \\ .... \\ 0..................(a_{mm}-\lambda) \end{bmatrix} = 0 \Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)......(a_{mm}-\lambda) = 0$$

$\Rightarrow \lambda = a_{11}, a_{22}, ...a_{mm}$ (diagonal elements of A) are the eigen values of A.

Exercise - 1.5

1. Write the characteristic matrices and the characteristic equations of the following matrices.

(a) $\begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$ **[Ans :** $\begin{bmatrix} 3-\lambda & 4 \\ 1 & 5-\lambda \end{bmatrix}$; $\lambda^2 - 8\lambda + 11 = 0$**]**

(b) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 4 & 5 \end{bmatrix}$ **[Ans :** $\begin{bmatrix} 1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 2 \\ 0 & 4 & 5-\lambda \end{bmatrix}$; $(1 - \lambda)(\lambda^2 - 8\lambda + 7) = 0$**]**

2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, find the eigen values of

(i) A^{-1} (ii) A^2 and (iii) A^3

[Ans : (i) $1, \frac{1}{4}, \frac{1}{6}$; **(ii)** 1, 16, 36; **(iii)** 1, 64, 216**]**

3. Find the product and sum of the eigen values of the following matrices without finding the eigen values.

(i) $\begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$ **[Ans :** (i) 8 ; 9; (ii) 60 ; 12**]**

4. Find the eigen values of the following matrices.

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$ **[Ans :** (i) 2, 2 , 2; (ii) $\frac{1}{2}\left(9 \pm \sqrt{41}\right)$**]**

5. Find the eigen values and the corresponding eigen vectors of the following matrices.

(i) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **[Ans :** $0, \pm\sqrt{2}$; $\begin{bmatrix} -c \\ 0 \\ c \end{bmatrix}, \begin{bmatrix} c \\ \sqrt{2}c \\ c \end{bmatrix}, \begin{bmatrix} -c \\ \sqrt{2}c \\ -c \end{bmatrix}$, $(c \neq 0)$**]**

(ii) $\begin{bmatrix} -2 & 8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ **[Ans :** 0, 1, 2 ; $\begin{bmatrix} 4c \\ -c \\ 0 \end{bmatrix}$; $\begin{bmatrix} 4c \\ 0 \\ -c \end{bmatrix}, \begin{bmatrix} 2c \\ -c \\ 0 \end{bmatrix}$, $c \neq 0$**]**

(iii) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ **[Ans :** 1, –1, 3 ; $\begin{bmatrix} c \\ -c \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ c \\ -c \end{bmatrix}, \begin{bmatrix} -2c \\ -3c \\ c \end{bmatrix}$, $c \neq 0$**]**

(iv) $\begin{bmatrix} 2 & 0 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 3 \end{bmatrix}$ **[Ans :** 2, 2, 4 ; $\begin{bmatrix} c \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} c \\ 0 \\ c \end{bmatrix}$, $c \neq 0$**]**

(v) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **[Ans :** 2, 2, 8 ; $\begin{bmatrix} c_1 \\ c_2 \\ c_2 - 2c_1 \end{bmatrix}, \begin{bmatrix} 2c \\ -c \\ c \end{bmatrix}$**]**

(vi) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **[Ans :** –3, –3, 5 ; $\begin{bmatrix} -2c_1 + 3c_2 \\ c_1 \\ c_2 \end{bmatrix}$; $\begin{bmatrix} c \\ 2c \\ -c \end{bmatrix}$**]**

1.8 Cayley-Hamilton Theorem

Cayley-Hamilton Theorem

It states that "Every square matrix satisfies its characteristic equation".

Proof : Let A be a square matrix of order n and I be the unit matrix of the same order.

Then $$|A - \lambda I| = 0 \qquad .....(1.53)$$

is the characteristic equation and 'λ' the characteristic root of A.

$\because$ (1.54) is an equation of degree 'n' in λ, it can be written as,

$$(-1)^n \left[\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_{n-1}\lambda + p_n\right] = 0 \qquad .....(1.54)$$

where $p_1, p_2, ..., p_n$ are constants.

$\therefore$ We have to prove that,

$$A^n + p_1A^{n-1} + p_2 A^{n-2} + ... + p_{n-1} A + p_n I = 0 \qquad .....(1.55)$$

The elements of the matrix $(A - \lambda I)$ are atmost of the first degree in λ.

$\therefore$ $Adj\ (A - \lambda I)$ has its elements as ordinary polynomials in λ of degree $(n - 1)$ or less. So $Adj\ (A - \lambda I)$ can be written as a matrix polynomial as,

$$Adj\ (A - \lambda I) = C_0\lambda^{n-1} + C_1\lambda^{n-2} + ... + C_{n-2}\lambda + C_{n-1},$$

where $C_0, C_2, ..., C_{n-1}$ are $n \times n$ matrices and their elements are functions of elements of A.

Again, since

$$(A - \lambda I) . Adj\ (A - \lambda I) = |A - \lambda I|I,$$

we have, $$(A - \lambda I)\left[C_0\lambda^{n-1} + C_1\lambda^{n-2} + \ldots + C_{n-2}\lambda + C_{n-1}\right]$$

$$= (-1)^n\left[\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_{n-1}\lambda + p_n\right]I$$

[from(1.54)]

Comparing coefficients of like powers of λ on both sides, we get

$$-IC_0 = (-1)^n I$$

$$AC_0 - IC_1 = (-1)^n . p_1 I$$

$$AC_1 - IC_2 = (-1)^n . p_2 I$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$AC_{n-2} - IC_{n-1} = (-1)^n . p_{n-1} I$$

$$AC_{n-1} = (-1)^n . p_n I$$

Premultiplying the above equations respectively by A^n, A^{n-1}, ... A, I and adding, we get

$$0 = (-1)^n \left[A^n + p_1 A^{n-1} + p_2 A^{n-2} + ... + p_{n-1} A + p_n I\right]$$

which is equal to (1.55) and hence the theorem is proved.

Corollary 1 :

The inverse of a nonsingular matirx can be found by using the above therom as follows :

A is nonsingular $\Rightarrow |A| \neq 0$.

Then premultiplying (1.55) of above theroem by A^{-1}, we get ;

$$A^{n-1} + p_1 A^{n-2} + ... + p_{n-1} I + p_n A^{-1} = 0$$

$$\Rightarrow \qquad A^{-1} = \frac{-1}{p_n}\left[A^{n-1} + p_1 A^{n-2} + ... + p_{n-1} I\right]$$

(If $\lambda = 0$ (1.53) and (1.54) $\Rightarrow |A| = (-1)^n p_n \Rightarrow p_n \neq 0$)

Corollary 2 :

If m be a positive integer such that $m \geq n$, then multiplying the result (1.55) by A^{m-n}, we get,

$$A^m + p_1 A^{m-1} + ... + p_n A^{m-n} = 0,$$

which shows that any positive integral power A^m $(m \geq n)$ of A can be expressed in terms of powers of lower order.

Solved Examples

***Example* 1.49:** Verify Cayley Hamilton Theorem for the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$.

Solution:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}; \; A - \lambda I = \begin{bmatrix} 3-\lambda & 2 \\ 1 & 5-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda) - 2$$

$$= \lambda^2 - 8\lambda + 13$$

characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \qquad \lambda^2 - 8\lambda + 13 = 0 \qquad .....(1.56)$$

Cayley Hamilton theorem states that A satisfies its characteristic equation.

i.e., We have to verify that $A^2 - 8A + 13I = 0$

$$A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3.3+2.1 & 3.2+2.5 \\ 1.3+5.1 & 1.2+5.5 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 16 \\ 8 & 27 \end{bmatrix}$$

$$A^2 - 8A + 13I = \begin{bmatrix} 11 & 16 \\ 8 & 27 \end{bmatrix} - 8\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} + 13\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11-24+13 & 16-16+0 \\ 8-8+0 & 27-40+13 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence the theorem is verified.

***Example* 1.50:** Velrify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$.

Solution:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(1-\lambda)\ (4-\lambda)-0] - 2[3(4-\lambda)-0]$$
$$-1[3+2(1-\lambda)]$$

$$= 4 - 9\lambda + 6\lambda^2 - \lambda^3 - 24 + 6\lambda - 5 + 2\lambda$$

$$= -\lambda^3 + 6\lambda^2 - \lambda - 25$$

Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - 6\lambda^2 + \lambda + 25 = 0$

Cayley-Hamilton theorem states that the matrix satisfies its characteristic equation.

$\therefore$ we have to verify that

$$A^3 - 6A^2 + A + 25I = 0$$

$$A^2 = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (1.1+2.3+-1.-2) & (1.2+2.1-1.1) & (1.-1+2.0-1.4) \\ (3.1+1.3+0.-2) & (3.2+1.1+0.1) & (3.-1+1.0+0.4) \\ (-2.1+1.3+4.-2) & (-2.2+1.1+4.1) & (-2-1+1.0+4.4) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 & -5 \\ 6 & 7 & -3 \\ -7 & 1 & 18 \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} 9 & 3 & -5 \\ 6 & 7 & -3 \\ -7 & 1 & 18 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 28 & 16 & -29 \\ 33 & 16 & -18 \\ -40 & 5 & 79 \end{bmatrix}$$

$$A^3 - 6A^2 + A + 25I = \begin{bmatrix} 28 & 16 & -29 \\ 33 & 16 & -18 \\ -40 & 5 & 79 \end{bmatrix} - 6\begin{bmatrix} 9 & 3 & -5 \\ 6 & 7 & -3 \\ -7 & 1 & 18 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} + 25\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (28-54+1+25) & (16-18+2+0) & (-29+30-1+0) \\ (33-36+3+0) & (16-42+1+25) & (-18+18+0+0) \\ (-40+42-2+0) & (5-6+1+0) & (79-108+4+25) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$\therefore$ The theorem is verified.

***Example* 1.51:** Using Cayley-Hamilton theorem find A^{-1} if $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 0 & -1 & 4 \end{bmatrix}$.

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & 3-\lambda & 5 \\ 0 & -1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2 - 7\lambda + 17) - 2(+8 - 2\lambda) + 1(-2) = 0$$

$$\Rightarrow \quad -\lambda^3 + 8\lambda^2 - 24\lambda + 17 - 16 + 4\lambda - 2 = 0$$

$$\Rightarrow \quad \lambda^3 - 8\lambda^2 - 20\lambda + 1 = 0$$

According to Cayley-Hamilton theorem,

$$A^3 - 8A^2 + 20A + 1I = O_{3\times3}$$

Multipying each term with A^{-1}, we get,

$$A^2 - 8A + 20I + 1A^{-1} = O_{3\times3}$$

$$\Rightarrow \quad A^{-1} = -\left[A^2 - 8A + 20I\right] \qquad .....(1.57)$$

$$A^2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 15 \\ 8 & 8 & 37 \\ -2 & -7 & 11 \end{bmatrix}$$

$$\therefore \quad A^{-1} = -\left[\begin{pmatrix} 5 & 7 & 15 \\ 8 & 8 & 37 \\ -2 & -7 & 11 \end{pmatrix} - 8\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 0 & -1 & 4 \end{pmatrix} + 20\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= -\begin{bmatrix} (5-8+20) & (7-16+0) & (15-8+0) \\ (8-16+0) & (8-24+20) & (37-40+0) \\ (-2-0+0) & (-7+8+0) & (11-32+20) \end{bmatrix}$$

$$= -\begin{bmatrix} 17 & -9 & 7 \\ -8 & 4 & -3 \\ -2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -17 & 9 & -7 \\ 8 & -4 & 3 \\ 2 & -1 & 1 \end{bmatrix}$$

***Example* 1.52:** Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$. Hence find A^{-1}.

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & -1 & 0 \\ 3 & 1-\lambda & -1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)(1-\lambda)(3-\lambda) + 3(3-\lambda) + 2 = 0$$

$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 11\lambda + 6 + 9 - 3\lambda + 2 = 0$$

$$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 14\lambda - 17 = 0$$

Cayley-Hamilton theorem states that A satisfies its characteristic equation.

i.e., we have to verify that

$$A^3 - 6A^2 + 14A - 17I = O_{3\times3} \qquad \text{.....(1.58)}$$

$$A^2 = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 7 & -2 & -4 \\ 10 & -2 & 9 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & -3 & 1 \\ 7 & -2 & -4 \\ 10 & -2 & 9 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -4 & 6 \\ 0 & -9 & -10 \\ 32 & -12 & 29 \end{bmatrix}$$

$\therefore \quad A^3 - 6A^2 + 14A - 17I$

$$= \begin{bmatrix} -5 & -4 & 6 \\ 0 & -9 & -10 \\ 32 & -12 & 29 \end{bmatrix} - 6\begin{bmatrix} 1 & -3 & 1 \\ 7 & -2 & -4 \\ 10 & -2 & 9 \end{bmatrix} + 14\begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} - 17\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-5-6+28-17) & (-4+18-14-0) & (6-6+0-0) \\ (0-42+42-0) & (-9+12+14-17) & (-10+24-14-0) \\ 32-60+28-0 & (-12+12+0-0) & (29-54+42-17) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence the theorem is verified.

To find A^{-1} : Multiply (1.58) by A^{-1} to get,

$$A^2 - 6A + 14I - 17A^{-1} = O_{3\times 3}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{17}\left[A^2 - 6A + 14I\right]$$

$$= \frac{1}{17}\left[\begin{pmatrix} 1 & -3 & 1 \\ 7 & -2 & -4 \\ 10 & -2 & 9 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & -1 \\ 2 & 0 & 3 \end{pmatrix} + 14\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{17}\begin{pmatrix} (1-12+14) & (-3+6+0) & (1-0+0) \\ (7-18+0) & (-2-6+14) & (-4+6+0) \\ (10-12+0) & (-2-0+0) & (9-18+14) \end{pmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 3 & 3 & 1 \\ -11 & 6 & 2 \\ -2 & -2 & 5 \end{bmatrix}$$

***Example* 1.53:** If $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$, (i) find its eigen values (ii) verify Cayley-Hamilton theorem for A. (iii) Find A^{-1} and (iv) Express $(A^5 - 4A^4 - 2A^3 - A^2 - I)$ as a linear polynomial in A.

Solution:

(i) Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - 5\lambda + 2 = 0 \qquad .....(1.59)$$

$\therefore$ Eigen values of A are roots of (1.59).

They are $\lambda = \dfrac{5+\sqrt{17}}{2}, \dfrac{5-\sqrt{17}}{2}$

(ii) Cayley Hamilton theorem states that a square matrix satisfies its characteristic equation.

i.e., we have to verify that A satisfies (1.59)

i.e., $$A^2 - 5A + 2I = O_{2 \times 2} \qquad .....(1.60)$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix}$$

$$\therefore (1.60) \Rightarrow A^2 - 5A + 2I = \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix} - 5\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-5+2 & 5-5+0 \\ 10-10+0 & 18-20+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence the theorem is verified.

(iii) From (1.60), we have

$$A^2 - 5A + 2I = O_{2 \times 2}$$

Multiplying by A^{-1}, we have,

$$A - 5I + 2A^{-1} = O_{2 \times 2}$$

$$\Rightarrow \qquad A^{-1} = \frac{1}{2}[-A+5I]$$

$$= \frac{1}{2}\left[\begin{pmatrix}-1 & -1\\ -2 & -4\end{pmatrix} + 5\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix}4 & -1\\ -2 & 1\end{bmatrix}$$

(iv) (1.60) $\Rightarrow$ $A^2 = 5A - 2I$ .....(1.61)

(1.61) × A $\Rightarrow A^3 = 5A^2 - 2A$

$= 5\,(5A - 2I) - 2A$ (from(1.61))

$= 23A - 10I$ .....(1.62)

(1.62) × A $\Rightarrow A^4 = 23A^2 - 10A$

$= 23\,(5A - 2I) - 10A$ (from(1.62))

$= 105A - 46I$ .....(1.63)

(1.63) × A $\Rightarrow A^5 = 105A^2 - 46A$

$= 105(5A - 2I) - 46A$ (from(1.63))

$= 479\,A - 210\,I$ .....(1.64)

Substituting the values of A^2, A^3, A^4, A^5 obtained above (in equations (1.61), (1.62), (1.63), (1.64) respectively) in the given expression, we get,

$$A^5 - 4A^4 - 2A^3 - A^2 - I = (479A - 210I) - 4(105A - 46I)$$
$$- 2\,(23A - 10I) - (5A - 2I) - I$$
$$= (8A - 5I)$$

***Example* 1.54:** If $A = \begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$, express A^4 in lower powers of A.

Solution: $A = \begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$;

The characteristic equation, $|A - \lambda I| = 0$

$\Rightarrow \qquad (2 - \lambda)^2 - 3 = 0$

$\Rightarrow \qquad \lambda^2 - 4\lambda + 1 = 0$

According to Cayley Hamilton theorem, we have,

$$A^2 - 4A + I = O_{2\times 2}$$

$$\Rightarrow \quad A^2 = 4A - I \qquad .....(1.65)$$

$$(1.65)\times A \Rightarrow \quad A^3 = 4A^2 - A \qquad .....(1.66)$$

$$(1.66)\times A \Rightarrow \quad A^4 = 4A^3 - A^2 \qquad .....(1.67)$$

(1.67) can be further simplified, if necessary, as

$$A^4 = 4(4A^2 - A) - A^2$$

$$= 15A^2 - 4A \qquad \text{from } (1.66)$$

$$= 15\,(4A - I) - 4A \qquad \text{from } (1.65)$$

$$= 56A - 15I$$

Exercise - 1.6

I. Verify Cayley–Hamilton theorem for the following matrices.

(i) $\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

II. Using Cayley Hamilton theorem, find the inverses of the followig matrices.

(i) $\begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$ **Ans :** $\frac{1}{3}\begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$ **Ans :** $\frac{1}{9}\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$ **Ans :** $\frac{1}{19}\begin{bmatrix} 6 & -5 & -4 \\ -3 & 12 & 2 \\ -1 & 4 & 7 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & -1 \\ 3 & 1 & 6 \end{bmatrix}$ **Ans :** $+\frac{1}{36}\begin{bmatrix} -1 & +14 & +3 \\ +15 & +6 & -9 \\ -2 & -8 & +6 \end{bmatrix}$

III Using Cayley – Hamilton theorem, express $(A^5 - 4A^3 + 3A^2 - 2A)$ as a linear polynomial in A, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

[Ans : 79A – 81I**]**

IV If $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$, express A^5 in powers of A of lower order.

[Ans : 149A – 385I**]**

V If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find the constants a, b, c, such that $A^3 = aA$, $A^4 = bA$ and $A^5 = cA$.

[Ans : a = b = c = 1**]**

VI If $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, prove that

(i) $A^4 = 4A^2$ and (ii) $A^5 = 16A$.

(**Hint :** Use Cayley – Hamilton theorem).

***Example* 1.55:** Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Hence find A^{-1}.

***Solution*:** Characteristic equation of matrix A is

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} = 0$$

or $(1 - \lambda)[(1 - \lambda)(-1 - \lambda) - 3] - 2(-1 - \lambda - 1) - 2(3 - 1 + \lambda) = 0$

or $(1 - \lambda)(-1 + \lambda^2 - 3) - 2(-\lambda - 2) - 2(2 + \lambda) = 0$

or $(1 - \lambda)(\lambda^2 - 4) + 2\lambda + 4 - 4 - 2\lambda = 0$

or $(-\lambda + 1)(\lambda^2 - 4) = 0$ or $\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$

By Cayley – Hamilton Theorem

$A^3 - A^2 - 4A + 4I = 0$

or $A^2 - A - 4I + 4A^{-1} = 0$

or $A^2 - A - 4I + 4A^{-1} = 0$ (Multiplying by A^{-1})

or $4A^{-1} = [-A^2 + A + 4I]$ .....(1.68)

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$

From (1.68), we have

$$4A^{-1} = -\begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1+1+4 & 2+2+0 & -2-2+0 \\ -3+1+0 & -6+1+4 & 2+1+0 \\ -3+1+0 & -2+3+0 & -2-1+4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$ **Ans.**

***Example* 1.56:** Find the characteristic equation of the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and verify that it is satisfied by A and hence obtain A^{-1}.

Express $A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$ in linear polynomial in A.

Solution: Characteristic equation is $|A - \lambda I| = 0$

$$\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} = 0$$

$(2 - \lambda)\,[(2 - \lambda)^2 - 1] + 1\,[-2 + \lambda + 1] + 1\,[1 - 2 + \lambda] = 0$

or $(2 - \lambda)^3 - (2 - \lambda) + \lambda - 1 + \lambda - 1 = 0$

or $(2 - \lambda)^3 - 2 + \lambda + \lambda - 1 + \lambda - 1 = 0$ or $(2 - \lambda)^3 + 3\lambda - 4 = 0$

or $8 - \lambda^3 - 12\lambda + 6\lambda^2 + 3\lambda - 4 = 0$

or $-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$ or $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

By Cayley – Hamilton Theorem $A^3 - 6A^2 + 9A - 4I = 0$ …..(1.69)

Verification: $A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{pmatrix} = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$A^3 - 6A^2 + 9A - 4I$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

So it is verified that the characteristic equation (1.69) is satisfied by A.

Inverse of Matrix A,

$$A^3 - 6A^2 + 9A - 4I = 0$$

On multiplying by A^{-1}, we get

$$A^2 - 6A + 9I - 4A^{-1} = 0 \text{ or } 4A^{-1} = A^2 - 6A + 9I$$

or $$4A^{-1} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{pmatrix}, \quad A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$ **Ans.**

$$A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$$

$$= A^3(A^3 - 6A^2 + 9A - 4I) + 2(A^3 - 6A^2 + 9A - 4I) + 5A - I$$

$$= 5A - I$$ **Ans.**

***Example* 1.57:** Find the characteristic equation of the matrix A

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

Hence find A^{-1}

Solution: Characteristic equation is

$$\begin{bmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)[1+\lambda^2 - 2\lambda + 4] - 3(2 - 2\lambda + 2) + 1(4 - 1 + \lambda) = 0$$

or $$(4-\lambda)(\lambda^2 - 2\lambda + 5) - 3(-2\lambda + 4) + (3 + \lambda) = 0$$

$$4\lambda^2 - 8\lambda + 20 - \lambda^3 + 2\lambda^2 - 5\lambda + 6\lambda - 12 + 3 + \lambda = 0$$

or $$-\lambda^3 + 6\lambda^2 - 6\lambda + 11 = 0 \text{ or } \lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$$

By Cayley Hamilton Theorem

$$A^3 - 6A^2 + 6A - 11I = 0$$

By multiplying by A^{-1} we get

$$A^2 - 6A + 6I - 11\,A^{-1} = 0 \text{ or } 11A^{-1} = A^2 - 6A + 6I$$

$$11A^{-1} = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} - 6\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} + \begin{bmatrix} -24 & -18 & -6 \\ -12 & -6 & 12 \\ -6 & -12 & -6 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix} \therefore A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$ **Ans.**

***Example* 1.58:** Find the characteristic equation of the matrix [CSVTU 2007]

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Verify Cayley-Hamilton theorem and hence evaluate the matrix equation.

$$A^8 - 5A^7 + 7\,A^6 - 3\,A^5 + A^4 = 5A^3 - 8\,A^2 + 2A - I$$

Solution: Characteristic equation of the matrix A is

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} = 0 \text{ or } (2-\lambda)[(1-\lambda)\,(2-\lambda)] - 1(0) + 1\,(0 - 1 + \lambda) = 0$$

or $\lambda^3 - 5\,\lambda^2 + 7\lambda - 3 = 0$

According to Cayley-Hamilton Theorem

$$A^3 - 5\,A^2 + 7A - 3I = 0 \qquad \text{.....(1.70)}$$

We have to verify the eq. (1.70).

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$A^3 - 5A^2 + 7A - 3I = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14-25+14-3 & 13-20+7+0 & 13-20+7+0 \\ 0+0+0+0 & 1-5+7-3 & 0-0+0-0 \\ 13-20+7+0 & 13-20+7-0 & 14-25+14-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence, Cayley Hamilton Theorem is verified.

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5 (A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$= A^5 \text{XO} + A\text{XO} + A^2 + A + I = +A^2 + A + I$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Exercise 1.7

1. Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Verify Cayley Hamilton Theorem for this matrix. Hence find A^{-1}.

Ans. $A^{-1} = \frac{1}{20}\begin{bmatrix} 7 & -2 & -3 \\ 1 & 4 & 1 \\ -2 & 2 & 8 \end{bmatrix}$

2. Use Cayley Hamilton Theorem to find the inverse of the matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Ans. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3. Using the Cayley-Hamilton Theorem, find A^{-1}, given that

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{bmatrix}$$

Ans. $-\frac{1}{5}\begin{bmatrix} 4 & -5 & -2 \\ 7 & -10 & -1 \\ -2 & 0 & 1 \end{bmatrix}$

4. Using Cayley Hamilton Theorem, find the inverse of the matrix

$$\begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$$

Ans. $\frac{1}{10}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 5 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

5. Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

and show that the equation is also satisfied by A. **Ans.** $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

6. Find the eign values of the matrix

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$$

Ans. Eigenvalues are 0, +1, –2.

7. Using Cayley-Hamilton relation obtain the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Ans. $\frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$

8. Show that the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

satisfies its characteristic equation. Hence find A^{-1}.

Ans. $\frac{1}{9}\begin{bmatrix} 7 & 2 & -10 \\ -2 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$

9. Use Cayley Hamilton Theorem to find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$$ **Ans.** $A^{-1} = \frac{1}{7}\begin{bmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{bmatrix}$

10. Verify Cayley-Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$ Hence evaluate A^{-1} **Ans.** $\frac{1}{11}\begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$

11. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, then express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.

Ans. $A + 5I$

12. If λ_1, λ_2 and λ_3 are the eigen values of the matrix

$\begin{bmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{bmatrix}$ then $\lambda_1 + \lambda_2 + \lambda_3$ is equal to

(i) –16 (ii) 2 (iii) – 6 (iv) –14 **Ans.** (ii)

13. The matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ is given. The eigen values of $4A^{-1} + 3A + 2l$ are

(A) 6, 15; (B) 9, 12 (C) 9, 15; (D) 7, 15 **Ans.** (C)

14. The matrix A is defined as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$$

Find the eigen values of $3A^3 + 5A^2 + 6A + 1$ **Ans.** 15, –15, –53

15. If a matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, find the matrix A^{32}, using Cayley Hamilton Theorem.