

Unit – I

Recapitulation of Mathematics

- Basics of Differentiation
- Rolle's and Lagrange's Theorem
- Tangent and Normal
- Indefinite and Definite Integral

CHAPTER – 1

Basics of Differentiation

1.1 Introduction

If $f(x)$ is a function of x , then $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ is defined as the differential coefficient of $f(x)$ at $x = a$, provided this limit exists and is finite.

The differential coefficient of $f(x)$ with respect to x is generally written as $\frac{d}{dx} f(x)$ or $f'(x)$ or $Df(x)$.

Note: Generally, $\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ is known as the first principle of differential calculus.

1.2 Differential Coefficient of a Function at a Point

The value of the derivative of $f(x)$ obtained by putting $x = a$ is called the differential coefficient of $f(x)$ at $x = a$ and is denoted by $f'(a)$ or $\left(\frac{dy}{dx}\right)_{x=a}$.

List of Formulae

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|--|--|--|
| 1. $\frac{d}{dx} x^n = nx^{n-1}$ | 2. $\frac{d}{dx} \log_e x = \frac{1}{x}$ | 3. $\frac{d}{dx} e^x = e^x$ |
| 4. $\frac{d}{dx} a^x = a^x \log a$ | 5. $\frac{d}{dx} x = 1$ | 6. $\frac{d}{dx} \sin x = \cos x$ |
| 7. $\frac{d}{dx} \cos x = -\sin x$ | 8. $\frac{d}{dx} \tan x = \sec^2 x$ | 9. $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$ |
| 10. $\frac{d}{dx} \sec x = \sec x \tan x$ | 11. $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$ | |
| 12. $\frac{d}{dx} C = 0$, where C is constant | | 13. $\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e$ |

Example 1: Differentiate $\frac{lx^2 + mx + n}{\sqrt{x}}$ w.r.t.x.

Solution: $y = \frac{lx^2 + mx + n}{\sqrt{x}}$

$$\begin{aligned}y &= lx^{\frac{3}{2}} + mx^{\frac{1}{2}} + nx^{-\frac{1}{2}} \\ \frac{dy}{dx} &= l \frac{d}{dx} x^{\frac{3}{2}} + m \frac{d}{dx} x^{\frac{1}{2}} + n \frac{d}{dx} x^{-\frac{1}{2}} \\ &= l \cdot \frac{3}{2} x^{\frac{3}{2}-1} + m \cdot \frac{1}{2} x^{\frac{1}{2}-1} + n \left(\frac{-1}{2} \right) x^{\frac{-1}{2}-1}\end{aligned}$$

$$\frac{dy}{dx} = \frac{3l}{2} x^{\frac{1}{2}} + \frac{m}{2} x^{\frac{-1}{2}} - \frac{n}{2} x^{\frac{-3}{2}}$$

1.3 Differentiation from First Principle

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Example 2: If $y = \sqrt{x}$, find $\frac{dy}{dx}$ by first principle

Solution: Given $f(x) = \sqrt{x}$

$$\begin{aligned}\text{by definition } \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}\end{aligned}$$

Example 3: Differentiate $\sin^2 x$ from first principle

Solution: Given $f(x) = \sin^2 x$, $f(x+h) = \sin^2(x+h)$

by definition

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\sin^2(x+h) - \sin^2 x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h+x) \sin(x+h-x)}{h} \\
 &\quad (\because \sin^2 A - \sin^2 B = \sin(A+B) \sin(A-B)) \\
 &= \lim_{h \rightarrow 0} \frac{\sin(2x+h) \sin h}{h} \\
 &= \lim_{h \rightarrow 0} (\sin 2x + h) \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= \lim_{h \rightarrow 0} \sin(2x + h) = \sin 2x
 \end{aligned}$$

(i) Differential coefficient of the product of two functions (product rule)

$$\frac{d}{dx}[f_1(x).f_2(x)] = f_1(x) \frac{d}{dx}f_2(x) + f_2(x) \frac{d}{dx}f_1(x)$$

(ii) Differential coefficient of the quotient of two functions (quotient rule)

$$\frac{d}{dx} \frac{f_1(x)}{f_2(x)} = \frac{f_2(x) \frac{d}{dx}f_1(x) - f_1(x) \frac{d}{dx}f_2(x)}{[f_2(x)]^2}$$

Example 4: Differentiate $x^2 \log x$ w.r.t. x .

$$\begin{aligned}
 \text{Solution: } \frac{d}{dx}(x^2 \cdot \log x) &= x^2 \frac{d}{dx} \log x + \log x \frac{d}{dx} x^2 \\
 &= x^2 \cdot \frac{1}{x} + \log x \cdot 2x \\
 &= x + 2x \log x \\
 &= x(1 + 2\log x)
 \end{aligned}$$

Example 5: If $y = \frac{x}{x+5}$, then prove that $x \frac{dy}{dx} = y(1-y)$.

$$\begin{aligned}
 \text{Solution: } y = \frac{x}{x+5} \text{ then } \frac{dy}{dx} &= \frac{(x+5)\frac{dy}{dx}x - x\frac{d}{dx}(x+5)}{(x+5)^2} \\
 &= \frac{(x+5) \cdot 1 - x(1+0)}{(x+5)^2} = \frac{(x+5)-x}{(x+5)^2}
 \end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{5}{(x+5)^2} \text{ then } x \frac{dy}{dx} = \frac{5x}{(x+5)^2} \\ &= \frac{x}{x+5} \cdot \frac{5}{x+5} = y \left(1 - \frac{x}{x+5}\right) \Rightarrow x \frac{dy}{dx} = y(1-y)\end{aligned}$$

1.4 Differential Coefficient of a Function of Function

If $y = f(z)$, where $z = F(x)$, then by eliminating z , we get $y = f(F(x))$

e.g., $\sin x^3$, $e^{\sin x}$, $\cos 2x$ are the function of function.

Again, let $y = f(t)$, where $t = g(x)$

Then $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$, this is known as '*chain rule*'.

Example 6: Differentiate $\log(\log \sin x)$ w.r.t. x .

Solution: Let $y = \log(\log \sin x)$

Put $\log \sin x = t$

$y = \log t$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{d}{dt} \log t \cdot \frac{d}{dx} \log \sin x \\ &= \frac{1}{t} \frac{d}{dx} \log \sin x\end{aligned}$$

Again, put $\sin x = u$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{t} \cdot \frac{d}{du} \log u \frac{du}{dx} \\ &= \frac{1}{t} \cdot \frac{1}{u} \cdot \frac{d}{dx} \sin x = \frac{1}{\log \sin x} \cdot \frac{1}{\sin x} \cdot \cos x \\ &= \frac{\cot x}{\log \sin x}\end{aligned}$$

Example 7: If $y = \log \sqrt{\frac{1-\cos mx}{1+\cos mx}}$, then find $\frac{dy}{dx}$.

Solution: $y = \log \sqrt{\frac{1-\cos mx}{1+\cos mx}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \log \sqrt{\frac{2\sin^2 \frac{mx}{2}}{2\cos^2 \frac{mx}{2}}} \\ &= \frac{d}{dx} \log \tan \frac{mx}{2}\end{aligned}$$

Put $\tan \frac{mx}{2} = t$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dt} \log t \cdot \frac{d}{dx} \tan \frac{mx}{2} \\ &= \frac{1}{t} \cdot \frac{d}{dx} \tan \frac{mx}{2}\end{aligned}$$

Again put $\frac{mx}{2} = u$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\tan \frac{mx}{2}} \cdot \frac{d}{du} \tan u \cdot \frac{d}{dx} \frac{mx}{2} \\ &= \frac{1}{\tan \frac{mx}{2}} \sec^2 u \cdot \frac{m}{2} \\ &= \frac{m}{2} \frac{1}{\tan \frac{mx}{2}} \cdot \sec^2 \frac{mx}{2} \\ &= \frac{m}{2} \cdot \frac{\cos \frac{mx}{2}}{\sin \frac{mx}{2}} \cdot \frac{1}{\cos^2 \frac{mx}{2}} \\ &= m \operatorname{cosec} mx\end{aligned}$$

1.5 Differential Coefficient of Inverse Trigonometric Functions or Trigonometrical Transformation

Formula 1. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

2. $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

3. $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

4. $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$

$$5. \frac{d}{dx} \sec^{-1} x = \frac{1}{x \sqrt{x^2 - 1}}$$

$$6. \frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{-1}{x \sqrt{x^2 - 1}}$$

Example 8: Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t. x.

Solution: Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan \theta \Rightarrow \theta = \tan^{-1} x$

$$\begin{aligned} y &= \sin^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right) \\ &= \sin^{-1} (\sin 2\theta) \quad \left(\because \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \end{aligned}$$

$$y = 2\theta, y = 2 \tan^{-1} x$$

$$\frac{dy}{dx} = 2 \frac{d}{dx} \tan^{-1} x = \frac{2}{1+x^2}$$

Example 9: If $y = \cot^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$, then find $\frac{dy}{dx}$.

Solution: $y = \cot^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$

Put $x = \tan \theta \Rightarrow \theta = \tan^{-1} x$

$$y = \cot^{-1}\left(\frac{\sqrt{1+\tan^2 \theta}+1}{\tan \theta}\right)$$

$$= \cot^{-1}\left(\frac{\sec \theta + 1}{\tan \theta}\right)$$

$$= \cot^{-1}\left(\frac{\frac{1}{\cos \theta} + 1}{\frac{\sin \theta}{\cos \theta}}\right)$$

$$= \cot^{-1}\left(\frac{1 + \cos \theta}{\sin \theta}\right)$$

$$\begin{aligned}
 &= \cot^{-1} \left(\frac{2 \cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right) \\
 &= \cot^{-1} \cot \frac{\theta}{2} \\
 y = \frac{\theta}{2} \Rightarrow \quad y &= \frac{1}{2} \tan^{-1} x \\
 \frac{dy}{dx} &= \frac{1}{2} \cdot \frac{1}{1+x^2}
 \end{aligned}$$

Example 10: Differentiate $\sin^{-1} \left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2} \right)$ w.r.t.x.

$$\begin{aligned}
 \text{Solution:} \quad \text{Let} \quad y &= \sin^{-1} \left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2} \right) \\
 \text{Put} \quad x &= \sin A, \sqrt{x} = \sin B \Rightarrow A = \sin^{-1} x, B = \sin^{-1} \sqrt{x} \\
 y &= \sin^{-1} \left(\sin A \sqrt{1-\sin^2 B} - \sin B \sqrt{1-\sin^2 A} \right) \\
 &= \sin^{-1} (\sin A \cos B - \cos A \sin B) \\
 &= \sin^{-1} \sin (A - B) \\
 y &= A - B \\
 y &= \sin^{-1} x - \sin^{-1} \sqrt{x} \\
 \frac{dy}{dx} &= \frac{d}{dx} \sin^{-1} x - \frac{d}{dx} \sin^{-1} \sqrt{x} \\
 &= \frac{1}{\sqrt{1-x^2}} - \frac{d}{dt} \sin^{-1} t \cdot \frac{dt}{dx} \quad \text{put } \sqrt{x} = t \\
 &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-t^2}} \cdot \frac{d}{dx} \sqrt{x} \\
 &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} \\
 &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}}
 \end{aligned}$$

1.6 Differentiation of Implicit Functions

There may be a relation between x and y when it is not possible to express it in the form of $y = f(x)$ 'or' to solve an equation for y in terms of x , relation is called implicit function.

For $\frac{dy}{dx}$ by differentiating the given relation w.r.t. x

Example 11: Find $\frac{dy}{dx}$, when $x^2 + y^2 = a^2$

Solution: Given $x^2 + y^2 = a^2$

Differentiating both the sides w.r.t. x , we get

$$\frac{d}{dx} x^2 + \frac{d}{dx} y^2 = \frac{d}{dx} a^2$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

Example 12: Find $\frac{dy}{dx}$, when $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Solution: Given $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Differentiating both the sides w.r.t. x , we get

$$a \frac{d}{dx} x^2 + 2h \frac{d}{dx} xy + b \frac{d}{dx} y^2 + 2g \frac{d}{dx} x + 2f \frac{d}{dx} y + \frac{d}{dx} c = 0$$

$$2ax + 2h \left(x \frac{dy}{dx} + y \frac{d}{dx} x \right) + 2by \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} + 0 = 0$$

$$2ax + 2hx \frac{dy}{dx} + 2yh + 2by \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} = 0$$

$$(2ax + 2hy + 2g) + (2hx + 2by + 2f) \frac{dy}{dx} = 0$$

$$2(hx + by + f) \frac{dy}{dx} = -2(ax + hy + g)$$

$$\frac{dy}{dx} = \frac{-(ax + hy + g)}{(hx + by + f)}$$

1.7 Logarithmic Differentiation

If the power of a function of x is also a function of x , then it would be convenient for us if we first take logarithms w.r.t the base e and then differentiate both sides w.r.t x .

Let us recall the formulae

- (i) $\log m^n = n \log m$
- (ii) $\log(mn) = \log m + \log n$
- (iii) $\log \frac{m}{n} = \log m - \log n$

Example 13: If $y = x^x$, then prove that $\frac{dy}{dx} = x^x(1 + \log x)$

Solution: Given $y = x^x$

Taking log on both the sides, we get

$$\begin{aligned}\log y &= \log x^x \\ \log y &= x \cdot \log x\end{aligned}$$

Differentiating both the sides w.r.t x , we get

$$\frac{d}{dx} \log y = x \frac{d}{dx} \log x + \log x \frac{d}{dx} x$$

$$\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\frac{1}{y} \frac{dy}{dx} = 1 + \log x$$

$$\frac{dy}{dx} = y(1 + \log x)$$

$$= x^x(1 + \log x)$$

1.8 Differentiation of Parametric Equation

When variables x and y are expressed in terms of a third variable like t or θ , then this third variable is called parameter.

e.g., $x = g(t)$ and $y = h(t)$, these are called parametric equations.

Formula
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example 14: If $x = at^2$, $y = 2at$, then find $\frac{dy}{dx}$.

Solution: Given $x = at^2$, $y = 2at$

$$\begin{aligned}\frac{dx}{dt} &= 2at, & \frac{dy}{dt} &= 2a \\ \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{1}{t}\end{aligned}$$

1.9 Differentiation of Infinite Series

When the function is given in infinite series. In this case we use the fact that if a term is deleted from an infinite series it remains unaffected.

Example 15: If $y = x^{x^{x^{x^{\dots^{\infty}}}}}$, find $\frac{dy}{dx}$.

Solution: Here $y = x^y$

$$\Rightarrow \log y = y \log x$$

Differentiating w.r.t x, we have

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= y \frac{d}{dx} \log x + \log x \frac{dy}{dx} \\ &= \frac{y}{x} + \log x \frac{dy}{dx} \\ \frac{dy}{dx} \left(\frac{1}{y} - \log x \right) &= \frac{y}{x} \\ \frac{dy}{dx} \left(\frac{1 - y \log x}{y} \right) &= \frac{y}{x} \\ \frac{dy}{dx} &= \frac{y^2}{x(1 - y \log x)}\end{aligned}$$

Example 16: If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots^{\infty}}}}$, Prove that $\frac{dy}{dx} = \frac{\cos x}{2y - 1}$

Solution: Here $y = \sqrt{\sin x + y}$

$$\begin{aligned}y^2 &= \sin x + y \\ 2y \frac{dy}{dx} &= \cos x + \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{\cos x}{2y - 1}\end{aligned}$$

Example 17: If $y = e^{x+e^{x+e^{x+\dots\infty}}}$, show that $\frac{dy}{dx} = \frac{y}{1-y}$

Solution: Here $y = e^{x+y}$

Taking log on both sides

$$\log y = (x + y) \log e \quad (\because \log e = 1)$$

$$\log y = x + y$$

Differentiating w.r.t x, we get

$$\frac{1}{y} \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} \left(\frac{1}{y} - 1 \right) = 1$$

$$\frac{dy}{dx} \left(\frac{1-y}{y} \right) = 1$$

$$\frac{dy}{dx} = \frac{y}{1-y}$$

Example 18: If $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots\infty}}}$ then prove that $\frac{dy}{dx} = \frac{1}{2y-1}$

Solution: Here $y = \sqrt{x+y}$ (1)

Taking log on both sides

$$\log y = \frac{1}{2} \log(x+y)$$

Differentiating w.r.t x, we get

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{1}{2} \cdot \frac{1}{x+y} \left(1 + \frac{dy}{dx} \right) \\ \left[\frac{1}{y} - \frac{1}{2(x+y)} \right] \frac{dy}{dx} &= \frac{1}{2(x+y)} \end{aligned} \quad \dots\dots(2)$$

From eq. (1) $x+y = y^2$

Putting in eq. (2) we get

$$\left(\frac{1}{y} - \frac{1}{2y^2} \right) \frac{dy}{dx} = \frac{1}{2y^2}$$

$$\left(\frac{2y-1}{2y^2} \right) \frac{dy}{dx} = \frac{1}{2y^2}$$

$$\frac{dy}{dx} = \frac{1}{2y-1}$$

1.10 Successive Differentiation

Differentiating the first derivative of a function we obtain second derivative, differentiating the second derivative, we get the third derivative and so on. This process of finding derivatives of higher order is known as successive differentiation.

Example 19: If $y = x + \tan x$, Prove that $\cos^2 x \frac{d^2y}{dx^2} + 2x = 2y$

Solution: Given $y = x + \tan x$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = 1 + \sec^2 x$$

Differentiating again, we get

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = 0 + 2 \sec x \cdot \sec x \tan x$$

$$\frac{d^2y}{dx^2} = 2 \sec^2 x \tan x$$

$$= \frac{2 \tan x}{\cos^2 x}$$

$$\cos^2 x \frac{d^2y}{dx^2} = 2 \tan x$$

$$\cos^2 x \frac{d^2y}{dx^2} = 2(y - x)$$

$$\cos^2 x \frac{d^2y}{dx^2} = 2y - 2x$$

$$\cos^2 x \frac{d^2y}{dx^2} + 2x = 2y$$

Example 20: If $y = ae^{mx} + be^{-mx}$, show that $\frac{d^2y}{dx^2} = m^2y$

Solution: Let $y = ae^{mx} + be^{-mx}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = am e^{mx} - bm e^{-mx}$$

differentiating again w.r.t. x, we have

$$\begin{aligned}\frac{d^2y}{dx^2} &= am^2 e^{mx} + bm^2 e^{-mx} \\ &= m^2 (ae^{mx} + be^{-mx}) \\ \frac{d^2y}{dx^2} &= m^2 y\end{aligned}$$

Example 21: If $y = A \cos nx + B \sin nx$, show that $\frac{d^2y}{dx^2} + n^2 y = 0$

Solution: Let $y = A \cos nx + B \sin nx$
differentiating w.r.t. x

$$\frac{dy}{dx} = -An \sin nx + Bn \cos nx$$

differentiating again w.r.t. x

$$\begin{aligned}\frac{d^2y}{dx^2} &= -An^2 \cos nx - Bn^2 \sin nx \\ &= -n^2 (A \cos nx + B \sin nx) \\ &= -n^2 y\end{aligned}$$

$$\frac{d^2y}{dx^2} + n^2 y = 0$$

Example 22: If $y = A \cos (\log x) + B \sin (\log x)$ prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

Solution: Let $y = A \cos (\log x) + B \sin (\log x)$

$$\begin{aligned}\text{differentiating w.r.t. } x, \frac{dy}{dx} &= -A \sin (\log x) \cdot \frac{1}{x} + B \cos (\log x) \cdot \frac{1}{x} \\ x \frac{dy}{dx} &= -A \sin (\log x) + B \cos (\log x)\end{aligned}$$

differentiating again w.r.t. x, we have

$$\begin{aligned}x \frac{d^2y}{dx^2} + \frac{dy}{dx} &= -A \cos(\log x) \cdot \frac{1}{x} - B \sin(\log x) \cdot \frac{1}{x} \\ x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} &= -[A \cos(\log x) + B \sin(\log x)] \\ &= -y\end{aligned}$$

$$\therefore x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$$

Example 23: If $y = Ae^{px} + Be^{qx}$ show that $\frac{d^2y}{dx^2} - (p+q) \frac{dy}{dx} + pqy = 0$

Solution: Let $y = Ae^{px} + Be^{qx}$
differentiating w.r.t. x,

$$\frac{dy}{dx} = Ape^{px} + Bqe^{qx} \quad \dots\dots(i)$$

differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = Ap^2e^{px} + Bq^2e^{qx} \quad \dots\dots(ii)$$

$$\begin{aligned} \text{Now } \frac{d^2y}{dx^2} - (p+q) \frac{dy}{dx} + pqy \\ &= Ap^2e^{px} + Bq^2e^{qx} - (p+q)(Ape^{px} + Bqe^{qx}) + pq(Ae^{px} + Be^{qx}) \\ &= Ap^2e^{px} + Bq^2e^{qx} - Ap^2e^{px} - Bq^2e^{qx} - Apqe^{px} - Bpqe^{qx} + pqAe^{px} + \\ &\quad pqBe^{qx} = 0 \end{aligned}$$

Practice Problems

Find the differential coefficient of the following

1. $\sqrt{x^{-3}}, 9x^{-8}$

2. $5x^3 - 4x^2 + 3x + 1$

3. $\frac{(x-4)(x+2)}{x}$

4. $\frac{(3x-1)(4x^{\frac{1}{2}}+5)}{x^{\frac{1}{2}}}$

5. $\frac{2x^3+6x^2+7}{\sqrt{x}}$

6. $x^5 - 2 \log x$

7. $e^x + x^n$

8. $(x \sin x + 7x^2 - 3) \cdot \frac{1}{x}$

9. $\frac{a+b\sqrt{x}}{\sqrt[3]{x}}$

10. $\frac{x e^x - 1}{x}$

11. $\frac{x^2 \log x - x + x^2 \cdot e^x}{x^2}$

12. $3e^x + 7 \log x + x^{-3}$

13. $e^{ax} \sin bx$

14. $3 \log x \sin x$

15. $2\sqrt{x} \log x$

16. $a e^x \sin x + b x^n \cos x$

17. $(x^2 + 1)(e^x + 2 \sin x)$
18. $\sqrt{x}(1-x)(x^2 + 2)$
19. $(1 - 2 \tan x)(5 + 4 \sin x)$
20. $\frac{2x^3 + 6x + 9}{x^2 + 5}$
21. $\frac{x^n}{\log_a x}$
22. $\frac{\sin x - x \cos x}{x \sin x + \cos x}$
23. $\frac{x \cdot \cos x}{\log x}$
24. $\frac{1 - \tan x}{1 + \tan x}$
25. $\frac{\sqrt{x} + \sqrt{a}}{\sqrt{a} - \sqrt{x}}$
26. $\frac{e^x + \tan x}{\cot x - x^n}$
27. $\frac{e^x + 5x^3 \log x}{\log x}$
28. $\frac{1}{(x+a)(x+b)(x+c)}$
29. $\frac{1 + \log x}{1 - \log x}$
30. $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$
31. $(2x - 3)\sqrt{7+x^2}$
32. $\sqrt{\frac{1-x}{1+x}}$
33. $\frac{1}{x + \sqrt{1+x^2}}$
34. $\frac{a^2 + x^2}{\sqrt{a^2 - x^2}}$
35. $\tan x^3$
36. $\log \tan x$
37. $\sin(\tan x)$
38. $\sqrt{\log x}$
39. e^{ax+b}
40. $\log \sqrt{\sin e^x}$
41. $\log\left(\frac{ax+b}{px+q}\right)$
42. $\frac{\sqrt{\sin x}}{\sin \sqrt{x}}$
43. $\cot(x^2 \sin 2x)$
44. $\sqrt{\frac{1+\sin x}{1-\sin x}}$
45. $\log \sqrt{\frac{1+\sin x}{1-\sin x}}$
46. $\log \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}}$
47. $\log(\sqrt{x+1} + \sqrt{x-1})$
48. $\sqrt{\operatorname{cosec} x \cot x}$
49. $\log \log \log x$
50. $\frac{e^{\sin x}}{\sin x^n}$
51. $\sqrt{\sec \sqrt{x}}$
52. If $y = \tan^{-1} \frac{\sqrt{1-x^2}}{x}$, find $\frac{dy}{dx}$
53. Differentiate $\sin^{-1}(3x - 4x^3)$

54. If $y = \tan^{-1} \sqrt{\frac{1-\cos x}{1+\cos x}}$, find $\frac{dy}{dx}$ differentiate the following:
55. $\sin^{-1} \frac{2x}{1+x^2}$
56. $\tan^{-1} \frac{x}{\sqrt{1+x^2}}$
57. $\sin^{-1} \left(\frac{1-x^2}{1+x^2} \right)$
58. $\cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$
59. $2 \tan^{-1} \sqrt{\frac{1-x}{1+x}}$
60. $\sin^{-1} \left(2x \sqrt{1-x^2} \right)$
61. $\cos^{-1} (4x^3 - 3x^2)$
62. $\tan^{-1} \left(\frac{a+x}{1-ax} \right)$
63. $\cos^{-1} \left(\frac{x-x^{-1}}{x+x^{-1}} \right)$
64. $\sin^{-1} \left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2} \right)$
65. If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ Then show that $\frac{dy}{dx} = y$
66. $\frac{3^x}{x + \tan x}$
67. $\sin x^o$ (Hint: $x^o = \frac{\pi}{180}$)

Find the Differential coefficient of the following functions by First Principle

68. x^7
69. $\cos(ax+b)$
70. $\tan ax$
71. $\cos^2 x$
72. $\cos^{-1} x$
73. e^{3x}
74. $\sin x^2$
75. $\cos \sqrt{x}$
76. $\tan 3x$
77. $x^{\frac{-3}{2}}$
78. If $\sin y = x \sin(a+y)$, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$
79. If $\frac{1}{2}(e^y - e^{-y}) = x$, prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$
80. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots \infty}}}$ prove that $(2y-1) \frac{dy}{dx} = \cos x$.
81. If $y \sqrt{1-x^2} + x \sqrt{1-y^2} = 1$, prove that $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1+x^2}} = 0$

Find $\frac{dy}{dx}$ for implicit differentiation when;

82. $xy = c^2$

83. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

84. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

85. $xy^3 - yx^3 = x$

86. $y \sec x + \tan x + x^2 y = 0$

87. $\tan(x+y) + \tan(x-y) = 1$

88. If $(x+y)^{m+n} = x^m y^n$, prove that $\frac{dy}{dx} = \frac{y}{x}$

89. Find the value of $\frac{dy}{dx}$ at $(2, -2)$ for the curve $x^2 + xy + y^2 = 4$

90. If $y = \sqrt{\log x} \sqrt{\log x + \sqrt{\log x + \dots \infty}}$ show that $(2y-1) \frac{dy}{dx} = \frac{1}{x}$

91. If $y = a^{x+y}$, show that $\frac{dy}{dx} = \frac{y \log a}{1 - y \log a}$

92. If $\cos(xy) = x$, show that $\frac{dy}{dx} + \frac{1+y \sin(xy)}{x \sin(xy)} = 0$

93. If $x = y \log(xy)$, find $\frac{dy}{dx}$

94. If $x^3 + y^3 = \sin(x+y)$, find $\frac{dy}{dx}$

95. If $x^3 y^2 = \log(x+y) + \sin e^x$, find $\frac{dy}{dx}$

96. If $x^a \cdot y^b = (x+y)^{a+b}$, prove that $\frac{dy}{dx} = \frac{y}{x}$ provided that $ay \neq bx$

Find $\frac{dy}{dx}$ of the following functions;

97. $(\log x)^x$

98. $(2x+3)^{x-5}$

99. $(\cos x)^{\log x}$

100. e^{x^x}

101. $(2-x) \left(\frac{3-x}{1+x} \right)^{\frac{1}{2}}$

102. $x^2 e^x \sin x$

103. $x \sin^{-1} x + (\log x)^x$

104. $(\tan x)^{\cot x} + (\cot x)^{\sin x}$

105. $\frac{1}{(x+a)(x+b)(x+c)}$

106. $\tan x \tan 2x \tan 3x \tan 4x$

107. x^{x^x}

108. $y = x^{1+x+x^2}$

109. $\frac{(x+1)(x+2)(x+3)}{(x-1)(x-2)(x-3)}$

110. $\frac{(x+2)^{\frac{5}{2}}}{(x+4)^{\frac{1}{3}}(x+1)^{\frac{3}{2}}}$ Find $\frac{dy}{dx}$ for the following parametric equations;

111. $x = a \sec^2 \theta, y = b \tan^2 \theta$

112. $x = a \sin^3 \theta, y = a \cos^3 \theta$

113. $x = \sqrt{\sin 2\theta}, y = \sqrt{\cos 2\theta}$

114. $x = at^4, y = bt^3$

115. $x = \frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3}$

116. $x = \sin^{-1} \frac{2u}{1+u^2}, y = \tan^{-1} \frac{2u}{1-u^2}$

117. $x = \log t, y = e^t + \cos t$

Find the derivative of the following functions

118. $\sin x^2$ w.r.t. x^2

119. $x^{\sin x}$ w.r.t. $(\sin x)^x$

120. $e^{\tan x}$ w.r.t. $\tan x$

121. $\tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}$ w.r.t. $\tan^{-1} x$

122. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = \frac{-y \log x}{x \log y}$

123. If $x = ae^t (\sin t + \cos t), y = ae^t (\sin t - \cos t)$ prove that $\frac{dy}{dx} = \tan t$

124. If $x = a \left(t + \frac{1}{t} \right)$ and $y = a \left(t - \frac{1}{t} \right)$ prove that $\frac{dy}{dx} = \frac{x}{y}$.

125. Find second derivative of $ax^3 + bx^2 + cx + d$

126. If $y = a \cos px + b \sin px$, prove that $\frac{d^2y}{dx^2} + p^2y = 0$

127. If $y = \log(1 + \cos x)$ then prove that $y_1 y_2 + y_3 = 0$

Hint: $y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2}, y_3 = \frac{d^3y}{dx^3}$

128. If $y = \tan x + \sec x$ then prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1 - \sin x)^2}$

129. If $y = x^2 \log x$ then prove that $\frac{d^3y}{dx^3} = \frac{2}{x}$

130. If $y = e^{ax} \sin bx$ then prove that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$

131. If $y = e^{m \cos^{-1} x}$ then prove that $(1-x^2)y_2 - xy_1 - m^2y = 0$

132. If $y = \sin(\sin x)$ then prove that $y_2 + y_1 \tan x + y \cos^2 x = 0$

133. If $y = t^2 - 1$, $x = 2t$ then prove that $\frac{d^2y}{dx^2} = \frac{1}{2}$
134. If $y = a \cos(\log x) + b \sin(\log x)$ then prove that $x^2y_2 + xy_1 + y = 0$
135. If $y = a \sin(\log x)$ then prove that $x^2y_2 + xy_1 + y = 0$
136. If $y = (\sin^{-1} x)^2$ then prove that $(1-x^2)\frac{d^2y}{dx^2} - x \frac{dy}{dx} = 2$
137. If $y = x + \tan x$ then prove that $\cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$
138. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ then prove that $(1-x^2)\frac{d^2y}{dx^2} - 3x \frac{dy}{dx} - y = 0$
139. If $y = Ae^{-kt} \cos(pt+c)$ then show that $\frac{d^2y}{dt^2} - 2k \frac{dy}{dt} - n^2y = 0$,
where $n^2 = p^2 + k^2$.
140. If $y = e^{\tan^{-1} x}$ then prove that $(1-x^2)y_2 - xy_1 - 2 = 0$
141. If $y = \frac{\log x}{x}$, then prove that $\frac{d^2y}{dx^2} + \frac{2\log x - 3}{x^3}$
142. If $y = \tan^{-1} x$ then prove that $\frac{d^2y}{dx^2} + \frac{2x}{(1+x^2)^2} = 0$
143. If $y = \sin(x+y)$ then prove that $\frac{d^2y}{dx^2} + y \left(1 + \frac{dy}{dx}\right)^3 = 0$
144. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then show that $\frac{d^2y}{dx^2} = \frac{-b^4}{a^2y^3}$.

Answers

1. $\frac{-3}{2}x^{\frac{-5}{2}}, -72x^{-9}$ 2. $15x^2 - 8x + 3$
3. $1 + \frac{8}{x^2}$ 4. $12 + \frac{15}{2}x^{\frac{-1}{2}} + \frac{5}{2}x^{\frac{-3}{2}}$
5. $\frac{5x^{\frac{3}{2}} + 9x^{\frac{1}{2}} - \frac{7}{2}}{2x^{\frac{3}{2}}}$ 6. $5x^4 - \frac{2}{x}$
7. $e^x + nx^{n-1}$ 8. $\cos x + 7 + \frac{3}{x^2}$

9. $\frac{-3}{2}ax^{\frac{-5}{2}} - \frac{b}{x^2}$
10. $e^x + \frac{1}{x^2}$
11. $\frac{x+x^2e^x+1}{x^2}$
12. $3e^x + \frac{7}{x} - 3x^{-4}$
13. $e^{ax}(a \sin bx + b \cos bx)$
14. $3\left(\frac{1}{x} \sin x + \cos x \log x\right)$
15. $\frac{1}{\sqrt{x}}(2 + \log x), \quad 16. ae^x(\cos x + \sin x) + bx^{n-1}(-x \sin x + n \cos x)$
17. $e^x(x^2 + 2x + 1) + 2 \cos x(x^2 + 1) + 4x \sin x$
18. $2x^{\frac{3}{2}}(1-x) + \frac{1}{2\sqrt{x}}(1-3x)(x^2+2)$
19. $4(\cos x - 2 \sin x) - 2 \sec^2 x(5 + 4 \sin x)$
20. $\frac{2(x^4 + 12x^2 - 9x + 15)}{(x^2 + 5)^2}$
21. $\tan \frac{x}{2} \sec^2 \frac{x}{2}$
22. $\frac{nx^{n-1} \log_a x - x^{n-1} \log_a e}{(\log_a x)^2}$
23. $\frac{x^2}{(x \sin x + \cos x)^2}$
24. $\frac{\log x (\cos x - x \sin x) - \cos x}{(\log x)^2}$
25. $\frac{-2}{1 + \sin 2x}$
26. $\frac{\sqrt{a}}{\sqrt{x}(\sqrt{a} - \sqrt{x})^2}$
27. $\frac{(e^x + \sec x^2)(\cot x - x^n) + (\operatorname{cosec}^2 x + nx^{n-1})(e^x + \tan x)}{(\cot x - x^n)^2}$
28. $\frac{(e^x + 15x^2 \log x + 5x^2) \log x - x^{-1}(e^x + 5x^3 \log x)}{(\log x)^2}$
29. $\frac{3x^2 + 2(a+b+c)x + (ab+bc+ca)}{(x+a)^2(x+b)^2(x+c)^2}$
30. $\frac{2}{x(1-\log x)^2}$
31. $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) \left(\frac{1}{\sqrt{x}} - \frac{1}{x\sqrt{x}}\right)$
32. $x(2x-3)(7+x^2)^{\frac{-1}{2}} + 2\sqrt{7+x^2}$
33. $-\frac{1}{(1+x)(1-x^2)^{\frac{1}{2}}}$
34. $-\frac{1}{(x+\sqrt{1+x^2})\sqrt{1+x^2}}$
35. $\frac{x(3a^2 - x^2)}{(a^2 - x^2)^{\frac{3}{2}}}$
36. $3x^2 \sec^2 x^3$
37. $\cot x \sec^2 x$
38. $\cos(\tan x) \sec^2 x$

39. $\frac{1}{2x\sqrt{\log x}}$

40. $a e^{ax+b}$

41. $\frac{e^x \cosec x}{2 \sin e^x}$

42. $\frac{a}{ax+b} - \frac{p}{px+q}$

43. $\frac{\sqrt{x} \cos x \sin \sqrt{x} - \sin x \cos \sqrt{x}}{2\sqrt{x}\sqrt{\sin x} \sin^2 \sqrt{x}}$

44. $-\cosec^2(x^2 \sin 2x)(2x \sin 2x + 2x^2 \cos 2x)$

45. $\frac{1}{1-\sin x}$

46. $\sec x$

47. $\frac{2}{\sqrt{x^2-1}}$

48. $\frac{1}{2\sqrt{x^2-1}}$

49. $\frac{-\cosec x (\cosec^2 x + \cot^2 x)}{2\sqrt{\cosec x \cdot \cot x}}$

50. $\frac{1}{x \log x \cdot \log \log x}$

51. $\frac{\sin x^n \cdot e^{\sin x} \cdot \cos x - e^{\sin x} \cdot \cos x^n \cdot nx^{n-1}}{\sin^2 x^n}$

52. $\frac{\sec \sqrt{x} \tan \sqrt{x}}{4\sqrt{x} \sqrt{\sec \sqrt{x}}}$

53. $\frac{-1}{\sqrt{1-x^2}}$

54. $\frac{3}{\sqrt{1-x^2}}$

55. $\frac{1}{2}$

56. $\frac{2}{1+x^2}$

57. $\frac{1}{(1+2x^2)\sqrt{1+x^2}}$

58. $\frac{-2}{1+x^2}$

59. $\frac{2}{1+x^2}$

60. $\frac{-1}{\sqrt{1-x^2}}$

61. $\frac{2}{\sqrt{1+x^2}}$

62. $\frac{-3}{\sqrt{1-x^2}}$

63. $\frac{1}{1+x^2}$

64. $\frac{-2}{1+x^2}$

65. $\frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}}$

67. $\frac{3^x [(x+\tan x) \log 3 - (1+\sec^2 x)]}{(x+\tan x)^2}$

68. $\frac{\pi}{180} \cos x^\circ$

69. $7x^6$

70. $-a \sin(ax+b)$

71. $a \sec^2 ax$

72. $-\sin 2x$
73. $\frac{-1}{\sqrt{1-x^2}}$
74. $3e^{3x}$
75. $2x \cos x^2$
76. $\frac{-\sin \sqrt{x}}{2\sqrt{x}}$
77. $3 \sec^2 3x$
78. $\frac{-3}{2} x^{-\frac{5}{2}}$
83. $\frac{-y}{x}$
84. $\frac{-b^2}{a^2} \cdot \frac{x}{y}$
85. $-\left(\frac{y}{x}\right)^{\frac{1}{3}}$
86. $\frac{1+3x^2y-y^3}{3xy^2-x^3}$
87. $\frac{-y \sec x \tan x + \sec^2 x + 2xy}{(\sec x + x^2)}$
88. $\frac{\sec^2(x+y) + \sec^2(x-y)}{\sec^2(x-y) - \sec^2(x+y)}$
90. $\frac{-2x+y}{x+2y}, 1$
94. $\frac{x-y}{x(1+\log xy)}$
95. $\frac{\cos(x+y)-3x^2}{3y^2-\cos(x+y)}$
96. $\frac{\frac{1}{x+y} + e^x \cosec x - 3x^2y^2}{2x^3y - \frac{1}{x+y}}$
98. $(\log x)^x \left[\log \log x + \frac{1}{\log x} \right]$
99. $(2x+3)^{x-5} \left[\frac{2(x-5)}{2x+3} + \log(2x+3) \right]$
100. $(\cos x)^{\log x} \left[\frac{1}{x} \cdot \log \cos x - \tan x \log x \right]$
101. $e^{x^x} \cdot x^x (1 + \log x)$
102. $\frac{x^2 - 7}{(1+x)^{\frac{3}{2}} \sqrt{3-x}}$
103. $x^2 e^x \sin x \left[\frac{2}{x} + 1 + \cot x \right]$
104. $x^{\sin^{-1}x} \left[\frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}} \right] + (\log x)^x \left[\frac{1}{\log x} + \log \log x \right]$
105. $(\tan x)^{\cot x} \cosec^2 x (1 - \log \tan x) + (\cot x)^{\tan x} \cdot \sec^2 x (\log \cot x - 1)$
106. $\frac{-1}{(x+a)(x+b)(x+c)} \left[\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c} \right]$
107. $2 \tan x \tan 2x \tan 3x \tan 4x (\cosec 2x + 2 \cosec 4x + 3 \cosec 6x + 4 \cosec 8x)$

108. $x^{x^x} \left[x^x (1 + \log x \log x + x^{x-1}) \right]$

109. $x^{1+x+x^2} \left[(2x+1) \log x + \frac{1}{x} + 1 + x \right]$

110. $\frac{(x+1)(x+2)(x+3)}{(x-1)(x-2)(x-3)} \left[\frac{2}{1-x^2} + \frac{4}{4-x^2} + \frac{6}{9-x^2} \right]$

111. $\frac{(x+2)^{\frac{5}{2}}}{(x+4)^{\frac{1}{3}}(x+1)^{\frac{3}{2}}} \left[\frac{5}{2(x+2)} - \frac{1}{3(x+4)} - \frac{3}{2(x+1)} \right]$

112. $\frac{b}{a}$ 113. $- \cot \theta$

114. $-(\tan 2\theta)^{\frac{3}{2}}$ 115. $\frac{3b}{4at}$

116. $\frac{t(2-t^3)}{1-2t^3}$ 117. 1

118. $t(e^t - \sin t)$ 119. $\cos x^2$

120. $\frac{x^{\sin x} \left(\frac{\sin x}{x} + \cos x \log x \right)}{(\sin x)^x (x \cot x + \log \sin x)}$ 121. $\frac{e^{\tan x}}{\cos^3 x}$

122. $\frac{1}{2}$ 126. $6ax + 2$