CHAPTER - 1

Propositional Calculus

Introduction

Logic means reasoning. One of the important aims of logic is to provide rules through which one can determine the validity of any particular argument. Logic concern with all types of reasonings such as legal arguments, mathematical proofs, conclusions in a scientific theory based upon a set of given hypothesis. The rules are called as rules of inference. The rules should be independent of any particular argument or discipline or language used in the argument.

We shall mean, by formal logic, a system of rules and procedures used to decide whether or not a statement follows from some given set of statements.

In order to avoid ambiguity, we use symbols. The symbols are easy to write and easy to manipulate. Hence, the logic that we study is named as "Symbolic logic". For example, observe the following statements.

(i) All men are mortal

(ii) Socrates is a man

Therefore (iii) Socrates is mortal

According to the logic, if any three statements have the following form

(i) All M are P (ii) S is M

Therefore (iii) S is P

then (iii) follows from (i) and (ii). The argument is correct, no matter whether the meanings of statements (i), (ii), and (iii) are correct. All that required is that they have the forms (i), (ii), and (iii). In Aristotelian logic, an argument of this type is called **syllogism**.

LEARNING OBJECTIVES

- *to Construct compound statements by using the connectives*
- *to identify the Tautologies and Contradictions*
- *to Construct the Truth Tables for the given expressions*
- *to find the Equivalent statements for a given expression*
- *to formulate DNF and CNF for the given expressions*

The formulation of the syllogism is contained in Aristotle's organon. It had a great fascination for medieval logicians, for almost all their work centered about ascertaining its valid moods. The three characteristic properties of a syllogism are as follows:

(i) It consists of three statements. The first two statements are called as **premises**, and the third statement is called as **conclusion**. The third one (**conclusion**) being a logical consequence of the first two (the **premises**).

(ii) Each of the three sentences has one of the four forms given in the Table.

Classification	Examples
Universal and affirmative judgment	All *X* is *Y* All monkeys are tree climbers All integers are real numbers All men are mortal
Universal and negative judgment	No *X* is *Y* No man is mortal No monkey is a tree climber No negative number is a positive number
Particular and affirmative judgment	Some *X* is *Y* Some men are mortal Some monkeys are tree climbers Some real numbers are integers
Particular and negative judgment	Some *X* is not *Y* Some men are not mortal Some monkeys are not tree climbers Some real numbers are not integers

So a **syllogism** is an argument consisting of two propositions called **premises** and a third proposition called the **conclusion**.

1.1 Statements and Notations

In any language, a sentence, in practice, is constructed by means of words. So a meaningful sequence of words is a sentence.

A statement is a sentence for which we can say whether it is true or false.

Consider the two sentences

(i) "Rama is a man"

(ii) "open the door"

We can say that "Rama is a man" is true. Hence it is a sentence.

It is clear that "open the door" is not a statement. Thus "open the door" is a sentence which is not a statement.

Example 1.1

Let us consider the following sentences.

(i) Socrates is a man
(ii) Rama killed Ravana
(iii) Open the door
(iv) $\sqrt{3}$ is an irrational number
(v) How are you?
(vi) Delhi is the Capital of India
(vii) Guntur is the Capital of Andhra Pradesh

The sentences (i), (ii), (iv), (vi), (vii) are statements. The statements (i), (ii), (iv), (vi) are true statements. The statement (vii) is false. The sentences (iii) and (v) are not statements.

Note:

"True" or "False" are called the truth value of the statement considered. "True" is denoted by "T" or "I". "False" is denoted by "F" or "O". For any given statement "P" we assign the truth value "T" or "I" if P is true. We assign the truth value "F" or "O" if P is false.

1.1.1 Subject and Predicate

Consider the statement "Rama is a man". In this statement "Rama" is the subject of the statement. The other part "is a man" is called predicate.

Note:

A simple statement is a statement that contains only one subject and one predicate. Simple statements are the basic units of our language to frame the rules of inference. These statements are also called as primary statements. These are the statements that cannot be further broken down into smaller statements. Such statements are also called as atomic statements. "3 + 2 = 5" and "7 + 5 = 12" are two primary (or simple or atomic) statements.

1.1.2 Notation

Observe the following:

(i) p: Socrates is a man
(ii) q: Guntur is the Capital City of Andhra Pradesh

In statements (i), (ii), p and q are the symbols used. Here "p" is a statement in symbolic logic that corresponds to the English statement "Socrates is a man".

Here "Socrates" is the subject and "is a man" is the predicate. The statement P (that is, "Socrates is a man") contains only one subject and only one predicate. Hence it is a primary statement.

Similarly the statement "q" that represents the English statement "Guntur is the Capital City of Andhra Pradesh" is a primary statement. Note that the symbols "p" and "q" were used as the names of the statements. We use this symbolic notation for statements throughout this unit.

From the above discussion, we understand that the basic unit of our objective language is called as a primary statement (or variable). We assume that these primary statements cannot be further broken down or analyzed into simple statements.

Example 1.2

Determine the truth value of each of the following statements:

(i) 6 + 2 = 7 and 4 + 4 = 8

(ii) Four is even

(iii) 4 + 3 = 7 and 6 + 2 = 8

[JNTU–H, Nov 2010 (Set–1)]

Solution:

(i) As 6 + 2 = 8, the statement 6 + 2 = 7 has truth value "False".
4 + 4 = 8 is true.

(ii) The truth value of "Four is even" is true.

(iii) The truth value of the statement "4 + 3 = 7" is true. The truth value of the statement "6 + 2 = 8" is true.

1.2 Connectives and Truth Tables

By using connectives like "not', "or", "and", we may combine two or more primary statements. The words "or", "and" are called as connectives.

Example 1.3

Consider two statements

(i) Rama is a boy.

(ii) Sita is a girl.

These two statements (i) and (ii) are primary statements. By using connective "and" we can combine these two statements. (iii) Rama is a boy and Sita is a girl.

The statement (iii) is called as compound statement.

The sentences constructed by using two or more primary (or simple) statements and certain sentential connectives are called as compound statements. The simple statements used to form compound statements are called as the components of the compound statement.

To form compound statement we use simple sentences and the connectives “and”, “or”, “if....then....”, “if and only if”, etc.

Example 1.4

Suppose that p and q are two statements given by

p: Rama is a boy

q: Sita is a girl

Here p and q are two simple statements. We may combine p and q to get different compound statements:

(i) p and q: Rama is a boy and Sita is a girl.

(ii) p or q: Rama is a boy or Sita is a girl.

(iii) If p then q: If Rama is a boy then Sita is a girl.

(iv) p if and only if q: “Rama is a boy” if and only if “Sita is a girl”.

1.2.1 Negation of a Statement

Associated with every given statement ‘p’ there is another statement called its negation. The negation of ‘p’ is nothing but “not p”. It is denoted by “~p” or “$\tilde{p}$” or $\bar{p}$ or “$\rceil$p”

It is important to understand that

If p is true then “~p” is false

If p is false then “~p” is true

Example 1.5

(i) The negation of

p: Socrates is a man is

~p: Socrates is not a man.

(ii) q: Delhi is the Capital of India

$\rceil$q (or ~q): Delhi is not the Capital of India

(iii) r: London is a City

$\rceil$r (or ~r): London is not a City

Note:

The truth value of a compound statement or negation of a statement depends on the truth values of the statements used to form a compound statement or to form a negation of a statement.

The depending of the truth values of the statements involved will be shown in the form of a table called "Truth Table".

Truth Table for negation is given below

p	~p
T	F
F	T

1.2.2 Conjunction (or meet)

Let p and q be two statements. The conjunction of the statements p, q is the statement "p and q" (denoted by $p \wedge q$). The truth value of $p \wedge q$ is true only if "p is true" and "q is true". In all other cases $p \wedge q$ has the truth value "False" (or F).

The truth table for "Conjunction" is given below

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth table for $p \wedge q$

The symbol " $\wedge$ "may be read as "meet". The meaning of the Truth table for $p \wedge q$ is as follows:

(i) If p is true and q is true then $p \wedge q$ is true;

(ii) If p is true and q is false then $p \wedge q$ is false;

(iii) If p is false and q is true then $p \wedge q$ is false;

(iv) If p is false and q is false then $p \wedge q$ is false.

Example 1.6

Translate the statement

"Rama and Mallikarjun went up the hill"into symbolic form.

Solution: suppose that

p: Rama went up to the hill

q: Mallikarjun went upto the hill

Then $p \wedge q$: Rama went upto the hill and Mallikarjun went upto hill.

Therefore the statement "Rama and Mallikarjun went up the hill" is symbolised as "$p \wedge q$".

1.2.3 Disjunction (or join)

Let p and q be two statements. The disjunction of the statements p, q is the statement "p or q" (denoted by $p \vee q$). The truth value of $p \vee q$ is false only if "p is false" and "q is false". In all other cases $p \vee q$ is true. The truth table for "Disjunction" is given below:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth table for $p \vee q$

The symbol "$\vee$" may be read as "join".

If p stands for 'I shall purchase a book' and q stands for 'I shall purchase a pencil, then $p \vee q$ stands for the compound statement "I shall purchase a book and a pencil".

The meaning of the truth table for $p \vee q$ is as follows:

(i) If p is true and q is true then $p \vee q$ is true

(ii) If p is true and q is false then $p \vee q$ is true

(iii) If p is false and q is true then $p \vee q$ is true

(iv) If p is false and q is false then $p \vee q$ is false

Example 1.7

Translate the statement

"Rama or Mallikarjun went up the hill" into symbolic form.

Solution: Suppose that

p: Rama went up to the hill

q: Mallikarjun went upto hill

Then $p \vee q$: Rama went upto the hill or Mallikarjun went upto the hill.

Therefore the statement "Rama or Mallikarjun went upto hill" is symbolised as "$p \vee q$".

Example 1.8

Write the following statements in symbolic form

(i) Anil and Sunil are rich

(ii) Neither Ramu nor Raju is poor

(iii) It is not true that Ravi and Raju are both rich

[JNTU–H, Nov 2010, Set–1]

Solution:

(i) Suppose that

p: Anil is rich

q: Sunil is Rich

Then the given statement can be written in the symbolic form as $p \wedge q$.

(ii) write

p: Ramu is poor

q: Raju is poor

~p: Ramu is not poor

~q: Raju is not poor

The other form of the given statement is "Ramu is not poor" and "Raju is not poor", so the answer is $(\sim p) \wedge (\sim q)$.

(iii) write

p: Ravi is rich

q: Raju is rich

$p \wedge q$: Ravi is rich and Raju is rich

$p \wedge q$: Both Ravi and Raju are rich

$\sim(p \wedge q)$: It is not true that both Ravi and Raju are rich.

Example 1.9

What is the negation of the statement:

"2 is even and –3 is negative"

[JNTU–H, Nov 2008 (Set–4)]

[JNTU–H, June 2010 (Set–2)]

Solution: Write

p: 2 is even

q: –3 is negative

Then $p \wedge q$: 2 is even and –3 is negative.

The negation of $p \wedge q$ is $\neg(p \wedge q)$

$\neg(p \wedge q) = (\neg p) \vee (\neg q)$ (by demorgan laws)

$\neg p$: 2 is not even

$\neg q$: –3 is not negative

Hence the negation of the given statement is $(\neg p) \vee (\neg q)$.

That is

2 is not even (or) –3 is not negative

Example 1.10

What is the compound statement that is true when exactly two of the three statements p, q and R are true?

[JNTU–H, June 2010(Set–2)]

[JNTU–H, Nov 2008 (Set–4)]

Solution: We note that

$(p \wedge q) \wedge (\neg r)$ is true if "p and q are true" and "r is false".

$(p \wedge r) \wedge (\neg q)$ is true if "p and r are true" and "q is false".

$(q \wedge r) \wedge (\neg p)$ is true if "q and r are true" and "p is false".

Hence $\left[(p \wedge q) \wedge (\neg r)\right] \vee \left[(p \wedge r) \wedge (\neg q)\right] \vee \left[(q \wedge r) \wedge (\neg p)\right]$ is the compound statement that is true when exactly two of three statements p, q and r are true.

1.2.4 Notation

For any two statements p and q

(i) The compound statement $\sim(p \wedge q)$ is denoted by "$p \uparrow q$"

(ii) The compound statement $\sim(p \vee q)$ is denoted by "$p \downarrow q$"

(iii) Note that the symbols "↑" and "↓" are also connectives.

1.2.5 Draw Truth Table for "p ↑ q"

Solution:

p	q	$p \wedge q$	$p \uparrow q$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Truth table for "p ↑ q"

1.2.6 Draw the Truth Table for "p ↓ q"

Solution:

p	q	$p \vee q$	$p \downarrow q$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Truth table for "p ↓ q"

1.2.7 Statement Formulas and Truth Tables

We know that simple (or primary or atomic) statements contains no connectives. Those statements which contains one or more simple statements and some connectives are called as compound (or composite or molecular) statements.

For example, let p and q be two simple statements. Then $\rceil p$, $p \wedge q$, $p \vee q$, $p \wedge (\rceil q)$, $(\rceil p) \vee (\rceil q)$ are some compound statements.

These compound statements are also called as statement formulas derived from the simple statements p and q. In this case, p and q are called as the components of the statement formulas. The truth value of a statement formula depend on the truth value of the primary statements involved in it.

Here $\rceil p$ means negation of p.
$\rceil (p \wedge q)$ means negation of $(p \wedge q)$

Example 1.11

Construct the truth tables for the following statement formulas

(i) ~(~p) [that is $\rceil(\rceil p)$]
(ii) $q \wedge (\rceil q)$
(iii) $p \vee (\rceil q)$
(iv) $(p \vee q) \vee \rceil p$

Solution:

(i)

p	~p	~(~p)
T	F	T
F	T	F

Truth table for ~(~p)

Note that the truth value of both p and ~(~p) are same in all cases

(ii)

q	⌉q	q∧(⌉q)
T	F	F
F	T	F

Truth table for q∧(⌉q)

(iii)

p	q	⌉q	p∨(⌉q)
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Truth table for p∨ (⌉q)

(iv)

p	q	p ∨ q	⌉p	(p ∨ q) ∨ ⌉p
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

Truth table for (p ∨ q) ∨ ⌉p

1.2.8 Implication (or Conditional Statement)

Let p and q be two statements. The implication of the statements p, q is a statement "if p, then q" (denoted by $p \rightarrow q$). The truth value of "$p \rightarrow q$"is false only if "p is true" and "q is false". In all other cases "$p \rightarrow q$" has truth value "true" (or T).

The truth table for "implication" is given below.

p	q	p → q
T	T	T
F	T	T
T	F	F
F	F	T

Truth table for p → q

(The symbol "$\rightarrow$" may be read as "conditional".) The meaning of the Truth table for $p \rightarrow q$ is as follows:

(i) If p is true and q is true then $p \rightarrow q$ is true

(ii) If p is false and q is true then $p \rightarrow q$ is true

(iii) If p is true and q is false then $p \rightarrow q$ is false

(iv) If p is false and q is false then $p \rightarrow q$ is true

Example 1.12

Construct the truth table for $(p \wedge \neg q) \rightarrow r$

Solution:

p	q	r	$p \wedge \neg q$	$(p \wedge \neg q) \rightarrow r$
T	T	T	F	T
T	T	F	F	T
T	F	T	T	T
T	F	F	T	F
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Truth table for $(p \wedge \neg q) \rightarrow r$

Example 1.13

Construct the truth table for $p \rightarrow p \vee q$

Solution:

p	q	$p \vee q$	$p \rightarrow p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Truth table for $p \rightarrow p \vee q$

1.2.9 Biconditional (or Double Implication)

Let p and q be two statements. The double implication of the statements p, q is a statement "p if and only if q" (denoted by $p \rightleftarrows q$). The truth value of $p \rightleftarrows q$ is true if "p" and "q" have the same truth values and if false if they have opposite truth values.

The truth table for "Double implication" is given below

p	q	$p \rightleftarrows q$
T	T	T
F	T	F
T	F	F
F	F	T

Truth table for $p \rightleftarrows q$

The symbol "$\rightleftarrows$" may be read as "if and only if". The meaning of the truth table for $p \rightleftarrows q$ is as follows:

(i) If "p" is true and "q" is true then $p \rightleftarrows q$ is true

(ii) If "p" is false and "q" is true then $p \rightleftarrows q$ is false

(iii) If "p" is true and "q" is false then $p \rightleftarrows q$ is false

(iv) If "p" is false and "q" is false then $p \rightleftarrows q$ is true

1.2.10 Construction of the Truth Table for $p \rightleftarrows q$

Solution: We know that $p \rightleftarrows q$ is nothing but $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$p \rightleftarrows q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Truth table for $p \rightleftarrows q$

1.2.11 Well Formed Formulas

We know that a statement formula is an expression which is a string consisting of variables, parentheses and connectives.

Now we provide a recursive definition for "statement formula". It is often called as a well-formed formula.

A well-formed formula can be generated by the following formula.

Rule-1: A statement symbol (or variable) is a well-formed formula

Rule-2: if x is a well-formed formula then ~x is a well formed formula

Rule-3: if x and y are well formed formulas then $(x \vee y)$, $(x \wedge y)$, $(x \rightarrow y)$, $(x \rightleftarrows y)$ are also well formed formulas.

A string consisting of statement symbols, parenthesis, connectives is a well formed formula if it can be obtained by finitely many applications of the rules 1, 2 and 3 mentioned above.

For example, p, $\neg p, p \vee q$, $p \wedge q$, $(\neg p) \vee q$, $p \rightarrow q$, $(p \rightarrow q) \rightarrow r$, $(p \rightarrow q) \vee (r \rightarrow q)$, $p \rightleftarrows (q \wedge r)$ are well formed formulas.

Example 1.14

Find the truth table for the propositional (statement) formula

$$(p \rightleftarrows \neg q) \rightleftarrows (q \rightarrow p)$$

[JNTU–H, Nov 2008 (set–4)]

[JNTU–H, June 2010 (set–2)]

Solution: The truth table for the given statement formula is given below. (as there are only two variables, the truth table consists exactly four rows).

p	**q**	**~q**	**$p \rightleftarrows \sim q$**	**$q \rightarrow p$**	**$(p \rightleftarrows \sim q) \rightleftarrows (q \rightarrow p)$**
T	T	F	F	T	F
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F

Example 1.15

Construct the truth table for the following statement $(\sim p \rightleftarrows q) \rightleftarrows (q \rightleftarrows r)$

[JNTU Nov 2010, Set–4]

Solution: The truth table for the given statement is as follows:

p	**q**	**r**	**~p**	**~q**	**$\sim p \rightleftarrows \sim q$**	**$q \rightleftarrows r$**	**$(\sim p \rightleftarrows \sim q) \rightleftarrows (q \rightleftarrows r)$**
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	T	F	F	T	F

Contd…

p	q	r	~p	~q	$\sim p \rightleftarrows \sim q$	$q \rightleftarrows r$	$(\sim p \rightleftarrows \sim q) \rightleftarrows (q \rightleftarrows r)$
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	F
F	F	F	T	T	T	T	T

Truth table for the expression $(\sim p \rightleftarrows \sim q) \rightleftarrows (q \rightleftarrows r)$

1.2.12 The Operation ⊕ or Δ

There is one more operation on statements frequently used denoted by $\oplus$ or Δ. It may be called as the operation "ring sum". The truth table for this operation $\oplus$ is given below

p	q	$p \oplus q$(or $p \,\Delta\, q$)
T	T	F
T	F	T
F	T	T
F	F	F

Example 1.16

If p and q are two statements, then show that the statement $(p \oplus q) \vee (p \downarrow q)$ is equivalent to $p \uparrow q$.

[JNTUH, June 2010, Set–4]

Solution: The equivalence of two compound statements is shown in the following truth table:

Truth Table

p (1)	q (2)	$p \oplus q$ (3)	$p \downarrow q$ (4)	$(p \oplus q) \vee (p \downarrow q)$ (5)	$p \uparrow q$ (6)
T	T	F	F	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	T	T	T

Since values in columns (5) and (6) are same, therefore the two given compound statements are equivalent.

Example 1.17

If p and q are two statements, then show that the statement $(p \uparrow q) \oplus (p \wedge q)$ is equivalent to $(p \vee q) \wedge (p \downarrow q)$.

Solution: The truth table of two compound statements is as follows:

Truth table

p (1)	q (2)	$p \uparrow q$ (3)	$p \wedge q$	$(p \uparrow q) \oplus (p \wedge q)$ (4)	$p \vee q$ (5)	$p \downarrow q$ (6)	$(p \vee q) \wedge (p \downarrow q)$ (7)
T	T	F		F	T	F	F
T	F	T		F	T	F	F
F	T	T		F	T	F	F
F	F	T		F	F	T	F

Since values in columns (4) and (7) are same, therefore the two given are equivalent.

1.3 Tautology and Contradiction

Tautology and contradiction are two important concepts in the study of logic.

1.3.1 Tautology

Tautology is an expression which has truth value 'T' for all possible values of the statement variables involved in the expression.

Example 1.18

Show that $p \vee (\sim p)$ is a tautology.

Solution:

p	~p	$p \vee (\sim p)$
T	F	T
F	T	T

Table for $p \vee (\urcorner p)$

Observe the Table for $p \vee (\urcorner p)$. It is clear that in all cases, the truth value of $p \vee (\urcorner p)$ is true. Hence $p \vee (\urcorner p)$ is a tautology.

Example 1.19

Prove that $(p \vee q) \vee (\urcorner p)$ is a tautology.

Solution:

p	q	$p \vee q$	~p	$(p \vee q) \vee (\sim p)$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

Truth table for $(p \vee q) \vee (\sim p)$

Observe the table for $(p \vee q) \vee (\sim p)$. It is clear that in all cases, the truth value of $(p \vee q) \vee (\sim p)$ is true. Hence $(p \vee q) \vee (\sim p)$ is a tautology.

Example 1.20

Show the implication: $\left[(p \to q) \to q\right] \to p \vee q$

[JNTU Nov 2008, Set no.3]

Solution: We have to show that "$\left[(p \to q) \to q\right] \to p \vee q$" is a tautology

p	q	$p \to q$	$(p \to q) \to q$	$p \vee q$	$\left[(p \to q) \to q\right] \to p \vee q$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	F	T	F	F	T

Truth table for $\left[(p \to q) \to q\right] \to p \vee q$

Observe the truth table for "$\left[(p \to q) \to q\right] \to p \vee q$". In all cases the truth value of the statement is T. Therefore given statement "$\left[(p \to q) \to q\right] \to p \vee q$" is a tautology.

Example 1.21

Show that the proposition is a tautology. $\left[(p \vee \sim q) \wedge (\sim p \vee \sim q\right] \vee q$

[JNTU Nov 2008, Set no.3]

Solution: The truth table for given statement is as follows:

Let $E = \left[(p \vee \sim q) \wedge (\sim p \vee \sim q\right] \vee q$

p	q	$\sim p$	$\sim q$	$(p \vee \sim q)$	$(\sim p \vee \sim q)$	$(p \vee \sim q) \wedge (\sim p \vee \sim q)$	E
T	T	F	F	T	F	F	T
T	F	F	T	T	T	T	T
F	T	T	F	F	T	F	T
F	F	T	T	T	T	T	T

Truth table for $\left[(p \vee \sim q) \wedge (\sim p \vee \sim q\right] \vee q$

Observe the truth table for the given statement $\left[(p \vee \sim q) \wedge (\sim p \vee \sim q)\right] \vee q$. In all the cases, the truth values of the statement are T. Therefore the given statement is a tautology.

Example 1.22

Determine which of the following is not a Tautology.

[JNTU June 2010 Set no.1]

(i) $\sim(p \to q) \to p$ (ii) $\sim(p \to q) \to \sim q$

(ii) $\sim p \wedge (p \vee q) \to q$ (iii) $(p \to q) \wedge (q \to p)$

Solution:

(i) The truth tables for the given statement " $\sim(p \to q) \to p$ " is as follows:

p	q	p → q	~ (p → q)	(~ (p → q) → p)
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

Truth table for $\sim(p \to q) \to p$

Observe the truth table for " $\sim(p \to q) \to p$ ". In all the cases, the truth value of the statement " $\sim(p \to q) \to p$ "is true. Therefore " $\sim(p \to q) \to p$ "is a tautology.

(ii) The truth table for the given statement " $\sim(p \to q) \to \sim q$ " is as follows:

p	q	~ q	p → q	~ (p → q)	~ (p → q) →~ q
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	T	F	T
F	F	T	T	F	T

Truth table for $\sim(p \to q) \to \sim q$

Observe the truth table for " $\sim(p \to q) \to \sim q$ ". In all cases, the truth value of the statement " $\sim(p \to q) \to \sim q$ "are T. Hence " $\sim(p \to q) \to \sim q$ " is a tautology.

(iii) The truth table for the given statement " $\sim p \wedge (p \vee q) \to q$ " is as follows

p	q	~ p	(p ∨ q)	~ p ∧ (p ∨ q)	~ p ∧ (p ∨ q) → q
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Truth table for $\sim p \wedge (p \vee q) \to q$

Observe the truth table for "$\sim p \wedge (p \vee q) \rightarrow q$". In all the cases, the truth values of the statement "$\sim p \wedge (p \vee q) \rightarrow q$" are T. Hence the statement "$\sim p \wedge (p \vee q) \rightarrow q$" is a tautology.

(iv) The truth table for the given statement "$(p \rightarrow q) \wedge (q \rightarrow p)$" is as follows:

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Truth table for $(p \rightarrow q) \wedge (q \rightarrow p)$

Observe the truth table for "$(p \rightarrow q) \wedge (q \rightarrow p)$". In all the cases, the truth values of the statement "$(p \rightarrow q) \wedge (q \rightarrow p)$" are not true. Hence the given statement "$(p \rightarrow q) \wedge (q \rightarrow p)$"is not a tautology.

1.3.2 Contradiction (or Fallacy)

A contradiction (or Fallacy) is an expression which has truth value 'F' for all possible values of the statement variables involved in that expression.

Example 1.23

Show that $p \wedge (\sim p)$ is a contradiction.

Solution:

p	~p	$p \wedge \sim p$
T	F	F
F	T	F

Truth table for $p \wedge \sim p$

Observe the truth table for "$p \wedge (\sim p)$". It is clear that in all cases, the truth value of "$p \wedge (\sim p)$" is false. Hence $p \wedge (\sim p)$ is a contradiction.

Example 1.24

Prove that $(p \wedge q) \rightleftarrows (\sim p \vee \sim q)$ is a contradiction

Solution:

p	q	$p \wedge q$	~p	~q	$(\sim p \vee \sim q)$	$p \wedge q \rightleftarrows (\sim p \vee \sim q)$
T	T	T	F	F	F	F
T	F	F	F	T	T	F
F	F	F	T	T	T	F
F	F	F	T	T	T	F

Truth table for $(p \wedge q) \rightleftarrows (\sim p \vee \sim q)$

Observe the truth table for "$(p \wedge q) \rightleftarrows (\sim p \vee \sim q)$". It is clear that in all cases, the truth value of "$(p \wedge q) \rightleftarrows (\sim p \vee \sim q)$" is false. Hence $(p \wedge q) \rightleftarrows (\sim p \vee \sim q)$ is a contradiction.

1.3.3 Contingency

A proposition (statement) that is neither a tautology nor a contradiction is called a contingency.

Example 1.25

Prove that the proposition (statement) "$(p \to q) \to (p \wedge q)$" is a contingency.

[JNTU Nov 2008 (Set–1)]

Solution: Given statement is $(p \to q) \to (p \wedge q)$.

The truth table of given statement "$(p \to q) \to (p \wedge q)$" is given below.

p	q	$p \to q$	$(p \wedge q)$	$(p \to q) \to (p \wedge q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Truth table for $(p \to q) \to (p \wedge q)$

Observe the truth table for the statement "$(p \to q) \to (p \wedge q)$". Clearly all the truth values of given statement is neither "T" nor "F". Therefore it is neither a tautology nor a contradiction. Hence the given statement is a contingency.

Example 1.26

Construct the truth table and show that $((p \wedge \sim q) \to r) \to (p \to (q \vee r))$ is a tautology.

Solution: Let E denote the given expression. First we form the truth table.

p	q	r	~q	$p \wedge \sim q$	$(p \wedge \sim q) \to r$	$p \to (q \vee r)$	E
T	T	T	F	F	T	T	T
T	T	F	F	F	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	F	F	T
F	T	T	F	F	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	F	T	T	T

From the truth table we conclude that E (the given statement formula) is a tautology

Example 1.27

(In this example, we denote the truth value T by '1' and the truth value F by '0'). Consider the statement q where p, q and r are three propositions.

Show that the given statement formula is a contingency.

Truth table for $(p \vee q) \wedge \bar{r}$

p	**q**	**r**	**$p \vee q$**	**$\bar{r}$**	**$(p \vee q) \wedge \bar{r}$**
0	0	0	0	1	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	1	0	0
1	0	0	1	1	1
1	0	1	1	0	0
1	1	0	1	1	1
1	1	1	1	0	0

From the truth table, we understand that $(p \vee q) \wedge \bar{r}$ is a contingency.

Example 1.28

Show that $[p \wedge (p \vee q)] \wedge \bar{p}$ is a contradiction.

Solution: Now we write down the truth table

p	**q**	**$p \vee q$**	**$p \wedge (p \vee q)$**	**$\bar{p}$**	**$[p \wedge (p \vee q)] \wedge \bar{p}$**
0	0	0	0	1	0
0	1	1	0	1	0
1	0	1	1	0	0
1	1	1	1	0	0

Observing the table, we can conclude that $[p \wedge (p \vee q)] \wedge \bar{p}$ is always false. Hence $[p \wedge (p \vee q)] \wedge \bar{p}$ is a contradiction.

Example 1.29

Show that $[p \wedge (p \vee q)] \vee \bar{p}$

Solution: Now we write down the truth table.

p	q	$p \vee q$	$p \wedge (p \vee q)$	$\overline{p}$	$[p \wedge (p \vee q)] \vee \overline{p}$
0	0	0	0	1	1
0	1	1	0	1	1
1	0	1	1	0	1
1	1	1	1	0	1

Observing the table we can conclude that $[p \wedge (p \vee q)] \vee \overline{p}$ is always true. Hence $[p \wedge (p \vee q)] \vee \overline{p}$ is a tautology.

Example 1.30

Show that the following statement is a tautology $\overline{p} \wedge (p \to q) \to \overline{q}$.

[JNTUH Nov 2010, Set No.2]

Solution: Given statement is $\overline{p} \wedge (p \to q) \to \overline{q}$

Now we construct the truth table for given statement

p	q	$\overline{p}$	$\overline{q}$	$p \to q$	$\overline{p} \wedge (p \to q)$	$\overline{p} \wedge (p \to q) \to \overline{q}$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Given statement is not a tautology.

1.4 Equivalence of Statements /Formulas

1.4.1 Statements/Formulas

Let A and B be two statements involving the variables p_1, p_2, ...p_n then we say A and B are equivalent if the truth value of A is equal to the truth value of B for every 2^n possible sets of truth values assigned to p_1, p_2, ...p_n and is denoted by $A \Leftrightarrow B$. In other words $A \Leftrightarrow B$ is a tautology.

Example 1.31

Verify that $(p \to q) \Leftrightarrow (\sim p \vee q)$

Solution: Given that $A : (p \to q)$, $B : (\sim p \vee q)$ are two statements. We have to verify $A \Leftrightarrow B$.

The truth table for given statements is given below.

p	**q**	**~ p**	**p → q**	**~ p ∨ q**
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Truth table for A & B

Observe the truth table for statements A & B. The truth values of A and B are equal in all cases. Therefore, the statement A is equivalent to statement B.

That is $A \Leftrightarrow B$ is a tautology.

Hence $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$

Example 1.32

Show that $\sim (\sim p)$ is equivalent to p.

Solution: Let A: p, B: $\sim (\sim p)$ be two statements.

The truth table for the statements A & B is given below

p	**~ p**	**~ (~ p)**
T	F	T
T	F	T
F	T	F
F	T	F

Truth table for statements p and $\sim (\sim p)$

Observe the truth table for statements p and $\sim (\sim p)$ truth values of statement p and statement $\sim (\sim p)$ are identical (or equal for all cases). Therefore statement p is equivalent to statement $\sim (\sim p)$.

1.4.2 Equivalent Formulas

1. $\left.\begin{array}{l} p \vee p \Leftrightarrow p \\ p \wedge p \Leftrightarrow p \end{array}\right\}$ Idempotent laws

2. $\left.\begin{array}{l} p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r \\ p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r \end{array}\right\}$ Associative laws

3. $\left.\begin{array}{l} p \vee q \Leftrightarrow q \vee p \\ p \wedge q \Leftrightarrow q \wedge p \end{array}\right\}$ Commutative laws

4. $\left.\begin{array}{l} p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \end{array}\right\}$ Distributive laws

5. $P \vee F \Leftrightarrow P$
   $P \wedge F \Leftrightarrow F$

6. $P \vee T \Leftrightarrow T$
   $P \wedge T \Leftrightarrow P$

7. $\left.\begin{array}{l} P \vee \sim P \Leftrightarrow T \\ P \wedge \sim P \Leftrightarrow F \end{array}\right\}$ Complement laws

8. $\left.\begin{array}{l} p \vee (p \wedge q) \Leftrightarrow p \\ p \wedge (p \vee q) \Leftrightarrow p \end{array}\right\}$ Absorption laws

9. $\left.\begin{array}{l} \sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q \\ \sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q \end{array}\right\}$ De Morgan's law

Example 1.33

Show that the following statements are logically equivalent (without using truth table).

$$\sim (p \vee (\sim p \wedge q) \Leftrightarrow (\sim p \wedge \sim q)$$

[JNTU Nov 2010 Set No.2]

Solution: Given statements are $A: \sim (p \vee (\sim p \wedge q)$, $B: (\sim p \wedge \sim q)$ we have to show the statements A and B are equivalent. Consider statement $A: \sim [p \vee (\sim p \wedge q)]$

$\Leftrightarrow \sim [(p \vee \sim p) \wedge (p \vee q)]$ [by distributive law]

$\Leftrightarrow \sim [T \wedge (p \vee q)]$ [since $p \vee \sim p$ is a tautology]

$\Leftrightarrow \sim (p \vee q)$ (law 5)

$\Leftrightarrow \sim p \wedge \sim q$ [by Demorgan's law]

Example 1.34

Show that the following statements are logically equivalent (without using truth table)

$$(p \rightarrow q) \wedge (p \rightarrow r) \Leftrightarrow p \rightarrow (q \wedge r)$$

[JNTU Nov. 2010 Set no.4]

Solution: Consider the statement

$(p \to q) \wedge (p \to r)$

$\Leftrightarrow (\sim p \vee q) \wedge (p \to r)$ [See example 1.4.2 since $(p \to q) \Leftrightarrow (\sim p \vee q)$]

$\Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee r)$ [since $(p \Rightarrow r) \Leftrightarrow (\sim p \vee r)$]

$\Leftrightarrow \left[\sim p \vee (q \wedge r)\right]$ [by Distributive law]

$\Leftrightarrow \left[p \to (q \wedge r)\right]$ [since $(p \to q) \Leftrightarrow (\sim p \vee q)$]

Therefore, the statements "$(p \to q) \wedge (p \to r)$" and "$p \to (q \wedge r)$" are equivalent.

Example 1.35

Show that the value of $\left[(p \downarrow p) \downarrow (q \downarrow q)\right]$ is logically equivalent to $p \wedge q$.

[JNTU June 2010 Set No.3]

Solution: We know that "$p \downarrow q$" is defined as "$\sim (p \vee q)$"

That is $(p \downarrow q) \Leftrightarrow \sim (p \vee q)$ .....(1.1)

Consider the statement

$\left[(p \downarrow p) \downarrow (q \downarrow q)\right] \Leftrightarrow \ \sim \left[(p \downarrow p) \vee (q \downarrow q)\right]$

$\Leftrightarrow \ \sim \left[\sim (p \vee p) \vee \sim (q \vee q)\right]$

(by the definition of "$\downarrow$")

$\Leftrightarrow \ \sim \left[\sim p \vee \sim q\right]$

($p \vee p \Leftrightarrow p$ and $q \vee q \Leftrightarrow q$)

$\Leftrightarrow \sim \left[\sim (p \wedge q)\right]$

[Demorgan law: $\sim (p \wedge q) = (\sim p \vee \sim q)$]

$\Leftrightarrow p \wedge q$

Therefore, the given statements $\left[(p \downarrow p) \downarrow (q \downarrow q)\right]$ and $p \wedge q$ are equivalent.

Example 1.36

Show that "$p \to q$" and "$\sim q \to \sim p$" are equivalent (use truth table).

Solution:

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Truth table for both "$p \to q$" and "$\sim q \to \sim p$"

Observe the truth table for "$p \to q$" and "$\sim q \to \sim p$". The truth values for both the statements "$p \to q$" and "$\sim q \to \sim p$" are identical in all cases. Hence "$p \to q$" and "$\sim q \to \sim p$" are equivalent statements.

Example 1.37

Let n be a find positive integer

Consider the statements

p: n is an even number

q: n + 1 is an odd number

prove that p and q are equivalent

Solution: Suppose that p is true. Then n is an even number. Since n is even, we know that (n + 1) is an odd number. Hence q is true.

Similarly if (n + 1) is an odd number then n = (n + 1) –1 is even. Hence $q \to p$ is true. Therefore $p \Leftrightarrow q$. That is p and q are two different equivalent statements.

Example 1.38

Prove that $\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$

Solution:

p	q	$\sim (p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Hence $\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$

1.5 Duality Law and Tautological Implication

1.5.1 Duality Law

Let A and B be any two formulas. Then A and B are said to be dual each other, if either one can be obtained from the other by replacing " $\wedge$ " by " $\vee$ " and " $\vee$ " by " $\wedge$ ".

Note:

(i) The connectives " $\wedge$ " and " $\vee$ " are dual each other.

(ii) The dual of the variable "T" is "F" and the dual of F is T.

Example 1.39

Write the dual of the following.

(i) $(p \vee q) \wedge r$

(ii) $(p \wedge q) \vee T$

(iii) $\sim (p \vee q) \wedge (p \vee \sim (q \wedge \sim s))$

Solution: The duals of the given formulas (statements) as follows:

(i) " $(p \wedge q) \vee r$ " is the dual of " $(p \vee q) \wedge r$ "

(ii) $(p \vee q) \wedge T$ is the dual of $(p \wedge q) \vee T$

(iii) $\sim (p \wedge q) \vee (p \wedge \sim (q \vee \sim s))$ is the dual of $\sim (p \vee q) \wedge (p \vee \sim (q \wedge \sim s))$

1.5.2 Tautological Implications

A statement A is said to tautologically imply a statement B if and only if $A \rightarrow B$ is a tautology.

We denote this fact by $A \Rightarrow B$ which is read as "A implies B".

Note:

(i) " $\Rightarrow$ " is not a connective, " $A \Rightarrow B$ " is not a statement formula.

(ii) $A \Rightarrow B$ states that " $A \rightarrow B$ is a tautology" or "A tautologically implies B".

The connectivities $\wedge$, $\vee$ and $\rightleftarrows$ are symmetric in the sense that

$p \wedge q \Leftrightarrow q \wedge p$

$p \vee q \Leftrightarrow q \vee p$ and

$p \rightleftarrows q \Leftrightarrow q \rightleftarrows p$ but $p \rightarrow q$ is not equivalent to $q \rightarrow p$.

1.5.3 Converse

For any statement formula $p \to q$, the statement formula $q \to p$ is called its converse.

1.5.4 Inverse

For any statement formula $p \to q$, the statement formula $\sim p \to \sim q$ is called its inverse.

1.5.5 Contrapositive

For any statement formula $p \to q$, the statement formula "$\sim q \to \sim p$" is called its Contrapositive.

Write down Contrapositive of the following statement.

If Rama have Rs.100/- he will spend Rs. 50/- for his friend Krishna.

Solution: write p: "Rama have Rs. 100/-"

q: "Rama spends Rs. 50/- for krishna"

Given statement is "$p \to q$".

The Contrapositive of "$p \to q$" is "$\sim q \to \sim p$".

Now $\sim q$: "Rama does not spend Rs.50/- for Krishna"

$\sim p$: Rama does not have Rs. 100/-

The required statement is as follows

If "Rama does not spend Rs. 50/- for Krishna" then "Rama does not have Rs. 100/-".

1.5.6 Some Implications

The following implications have important applications. All of them can be proved by truth table or by any other methods.

$$p \wedge q \Rightarrow p \quad .....(1.2)$$

$$p \vee q \Rightarrow q \quad .....(1.3)$$

$$p \Rightarrow (p \vee q) \quad .....(1.4)$$

$$\sim p \Rightarrow (p \to q) \quad .....(1.5)$$

$$Q \Rightarrow (p \to q) \quad .....(1.6)$$

$$\sim (p \to q) \Rightarrow p \quad .....(1.7)$$

$$\sim (p \to q) \Rightarrow \sim q \quad .....(1.8)$$

$$p \wedge (p \to q) \Rightarrow \sim p \qquad .....(1.9)$$

$$\sim p \wedge (p \vee q) \Rightarrow q \qquad .....(1.10)$$

$$(p \to q) \wedge (q \to r) \Rightarrow (p \to r) \qquad .....(1.11)$$

$$(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r \qquad .....(1.12)$$

$$(\sim q) \wedge (p \to q) \Rightarrow (\sim p) \qquad .....(1.13)$$

Example 1.40

Show that $(\sim q) \wedge (p \to q) \Rightarrow (\sim p)$

Solution: we have to show that the statement "$\sim q \wedge (p \to q)$" is tautological implication to statement "~ p"

Suppose that $\sim q \wedge (p \to q)$ has truth value "T". This means, both "~q" and "p→q"has the truth value T.

This shows that "q" has truth value "F" and $p \to q$ has truth value T. Since the truth value of q is F, by the implication (or conditional) table. We conclude that the truth value of p is F.

Therefore the truth value of "~ p" is T. Hence "$\sim q \wedge (p \to q) \Rightarrow \sim p$"

Example 1.41

Construct the truth tables of converse, inverse and Contrapositive of the proposition (p→q).

[JNTU, June 2010, Set No.3]

Solution: Given proposition is "$p \to q$"

(i) The converse of the given proposition is "q → p".
(ii) The inverse of the given proposition is "~ p → ~ q".
(iii) The Contrapositive of the given proposition is "~ q → ~ p".

(i) The truth table of the converse of the proposition "p → q" is as follows:

p	q	$(p \to q)$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Truth table for proposition "$q \to p$"

(Converse of proposition "$p \to q$")

(ii) The truth table for the inverse of the proposition " $p \to q$ " is as follows:

p	q	~p	~q	$p \to q$	$\sim p \to \sim q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Truth table for the proposition " $\sim p \to \sim q$ "

(Inverse proposition of proposition " $p \to q$ ")

(iii) The truth table for the Contrapositive of the proposition " $p \to q$ " is as follows:

p	q	~p	~q	$p \to q$	$\sim q \to \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Truth table for the proposition " $\sim q \to \sim p$ "

(Contrapositive of the proposition " $p \to q$ ")

1.6 Normal Forms

Suppose that A (P_1, P_2,...P_n) be a statement formula where P_1, P_2,...P_n are the atomic variables. Each P_i have truth value T (or 1) or F (or 0). Hence the n-table (P_1, P_2,...P_n) have 2^n values. We can form the truth table for A(P_1, P_2,...P_n) with 2^n rows.

If for all 2^n values of (P_1, P_2,...P_n) the value of A (P_1, P_2,...P_n) is T (or 1) then the statement formula is said to be identically true. In this case we also say that A (P_1, P_2,...P_n) is a tautology.

If for all 2^n values of (P_1, P_2,...P_n) the value of A(P_1, P_2,...P_n) is F (or 0) then the statement formula A(P_1, P_2,...P_n) is said to be identically false. In this case, we also say that A(P_1, P_2,...P_n) is a contradiction.

If the truth value of A(P_1, P_2,...P_n) is True (T or 1) for atleast one value of (P_1, P_2,...P_n) then A(P_1, P_2,...P_n) is said to be satisfiable.

1.6.1 Decision Problem

The problem of determining (in a finite number of steps) whether the given statement formula is a tautology (or) a contradiction (or) satisfiable is called as a decision problem.

So every problem in the statement calculus has a solution (we can decide this by forming a truth table for the given statement formula.

Now we study different forms called as normal forms

(i) Disjunctive Normal Form (DNF)

(ii) Conjunctive Normal Form (CNF)

(iii) Principal Disjunctive Normal Form (PDNF)

(iv) Principal Conjunctive Normal Form (PCNF)

1.6.2 DNF

Let $X_1, X_2, \ldots X_n$ be n given atomic variables and $\overline{X}_1, \overline{X}_2, \ldots \overline{X}_n$ (or $(\sim X_1, \sim X_2 \ldots \sim X_n)$ are the negations of $X_1, X_2, \ldots X_n$ respectively.

Product of same elements from $\{X_1, X_2 \ldots X_n, \overline{X}_1, \overline{X}_2 \ldots \overline{X}_n\}$ is called as an elementary product.

Sum of some elements from $\{X_1, X_2 \ldots X_n, \overline{X}_1, \overline{X}_2 \ldots \overline{X}_n\}$ is called as elementary sum.

For example, $X_i, \overline{X}_i \wedge X_j$, $\overline{X}_1 \wedge X_3X_4$, $X_1 \wedge X_2 \wedge \ldots \wedge X_n$, $X_1 \wedge X_2 \wedge \overline{X}_3 \wedge X_4 \wedge \overline{X}_5$ are elementary products.

$\overline{X}_1$, $X_1 \vee X_2$, $\overline{X}_3 \vee X_4 \vee X_5$, $X_1 \vee X_2 \vee \ldots \vee X_n$, $\overline{X}_1 \vee \overline{X}_2 \vee X_3 \vee \overline{X}_4 \vee X_5$ are elementary sums.

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called as Disjunctive Normal Form (DNF) of the given statement formula.

1.6.3 How to Find DNF

Suppose a statement formula $A(P_1, P_2, \ldots P_n)$ is given. If it is in the form: sum of elementary products then it is already in DNF. If it is not in the form of DNF then we use some known results or formulas or axioms. Most of the cases when '→' presents, then we use the known result.

$$P \rightarrow Q \Leftrightarrow \rceil P \vee Q. \text{ [that is } \overline{P} \vee Q \text{].}$$

We use distributive laws, demorgan laws, Commutative and associative laws, etc.

Example 1.42

Find DNF of $X \wedge (X \rightarrow Y)$

Solution:

$$X \wedge (X \rightarrow Y)$$

$$\Leftrightarrow X \wedge (\rceil X \vee Y) \qquad \text{[by a known result } X \rightarrow Y \Leftrightarrow \overline{X} \vee Y \text{]}$$

$$\Leftrightarrow \left[X \wedge (\rceil X)\right] \vee [X \wedge Y] \qquad \text{[by distributive law]}$$

Now $\left[X \wedge \rceil X\right] \vee [X \wedge Y]$ is the sum of two elementary product terms $X \wedge \rceil X$ and $X \wedge Y$. Hence $\left[X \wedge \rceil X\right] \vee [X \wedge Y]$ is the DNF of the given statement formula $X \wedge (X \rightarrow Y)$.

1.6.4 CNF

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called as Conjunctive Normal Form (CNF) of the given statement formula.

Example 1.43

Find CNF for the statement formula $X \wedge (X \rightarrow Y)$

Solution:

$$X \wedge (X \rightarrow Y)$$

$$\Leftrightarrow X \wedge \left(\overline{X} \vee Y\right)$$

The formula $X \wedge \left(\overline{X} \vee Y\right)$ is a product of sums: X and $\overline{X} \vee Y$. So it is in CNF. Hence $X \wedge \left(\overline{X} \vee Y\right)$ is a CNF for $X \wedge (X \rightarrow Y)$.

1.6.5 Principal Disjunctive Normal Form (PDNF)

Let $P_1, P_2, ..., P_n$ be n statement variables. The expression $P_1^* \wedge P_2^* \wedge ... \wedge P_n^*$ where P_i^* is either P_i or $\sim P_i$ is called a minterm. There are 2^n such minterms.

The express $P_1^* \vee P_2^* \vee ... \vee P_n^*$, P_i^* is either P_i or $\sim P_i$ is called a maxterm. There are 2^n such maxterms.

Let P, Q, R be the three variables.

Then the minterms are:

$$P \wedge Q \wedge R, P \wedge Q \wedge \sim R, P \wedge \sim Q \wedge R, P \wedge \sim \wedge \sim R, \sim P \wedge Q \wedge R, \sim P \wedge Q \wedge \sim R,$$

$$\sim P \wedge \sim Q \wedge R, \sim P \wedge \sim Q \wedge \sim R.$$

For a given formula, an equivalent formula consisting of disjunction's of minterms only is known as its Principal Disjunctive Normal Form (PDNF) or sum of products canonical form.

Example 1.44

Find PDNF for $(\overline{X} \vee Y)$.

Solution: $\overline{X} \vee Y \Leftrightarrow (\overline{X} \wedge 1) \vee (Y \wedge 1)$ (since A ∧ 1 = A for all A)

$\Leftrightarrow \left[\overline{X} \wedge (Y \vee \overline{Y})\right] \vee \left[Y \wedge (X \vee \overline{X})\right]$ (since $A \vee \overline{A} = 1$)

$\Leftrightarrow \left[(\overline{X} \wedge Y) \vee (\overline{X} \wedge \overline{Y})\right] \vee \left[(Y \wedge X) \vee (Y \wedge \overline{X})\right]$ (by distributive law)

$\Leftrightarrow (\overline{X} \wedge Y) \vee (\overline{X} \wedge \overline{Y}) \vee (X \wedge Y) \vee (\overline{X} \wedge Y)$

$\Leftrightarrow (\overline{X} \wedge Y) \vee (\overline{X} \wedge \overline{Y}) \vee (X \wedge Y)$ (since $A \vee A = A$)

Hence $(\overline{X} \wedge Y) \vee (\overline{X} \wedge \overline{Y}) \vee (X \wedge Y)$ is the PDNF for $(\overline{X} \wedge Y)$

The following notation will be used in the next coming Black box method of finding PDNF. For convenience often we are denoting ~p by $\overline{p}$.

We also use the following notation.

Suppose there are three atomic variables p, q, r. Then observe the following:

Binary Notation			Expression		
0	0	0	$\overline{p}$	$\overline{q}$	$\overline{r}$
0	0	1	$\overline{p}$	$\overline{q}$	r
0	1	0	$\overline{p}$	q	$\overline{r}$
0	1	1	$\overline{p}$	q	r
1	0	0	p	$\overline{q}$	$\overline{r}$
1	0	1	p	$\overline{q}$	r
1	1	0	p	q	$\overline{r}$
1	1	1	p	q	r

It is clear that $\overline{p}\,\overline{q}\,\overline{r}$ is the product term (or related expression) for 000; $pq\overline{r}$ is the related expression for 110.

1.6.6 Black Box Method (to find PDNF)

Suppose that $A(X_1, X_2, ...X_n)$ be the given statement formula where $X_1, X_2, ...X_n$ are atomic variables and each atomic variable may attain its value either 0 or 1 (that is, False or True).

Form the truth table for $A(X_1, X_2, ...X_n)$ that contains 2^n rows.

This truth table determines the PDNF simply by taking each product term that occurs when $A(X_1, X_2, ...X_n)$ takes Value 1.

Example 1.45

Find PDNF for the statement P→Q by Black Box Method (or by using truth tables)

Solution: First we form the truth table for P→Q.

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

Or

P	Q	P → Q
1	1	1
1	0	0
0	1	1
0	0	1

Consider the column under P→Q.

These are three 1's in this column.

The 1's are in 1st row, 3rd row and forth row. Consider 1st row in this row P & Q have truth values 1 and 1 respectively. So the related product term is PQ.

Consider 3rd row the truth values of P and Q are 0 and 1 respectively.

So the related product term is $\overline{P}.Q$. Consider 4th row the truth values of P and Q are 0 and 0 respectively. So the related product term is $\overline{P}\,\overline{Q}$.

The PDNF is $PQ \vee \overline{P}Q \vee \overline{P}\,\overline{Q}$

In other words $(P \wedge Q) \vee (\overline{P} \wedge Q) \vee (\overline{P} \wedge \overline{Q})$.

Example 1.46

Find PDNF for $(\overline{X} \vee Y)$ (by Black Box Method)

(Compare this problem with example 1.6.8)

Solution: First we form truth table for $(\overline{X} \vee Y)$.

X	Y	$\overline{X}$	$\overline{X} \vee Y$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

Truth table for $\overline{X} \vee Y$

Observe the column of $\left(\overline{X} \vee Y\right)$. These are three's the one's are in first row, second row and fourth row.

Consider row–1

The product term related to row–1 is $\overline{X}\,\overline{Y}$ (because the truth values of X and Y are 0 and 0 respectively)

Consider row–2

The product term related to row–2 is $\overline{X}\,Y$ (because the truth values of X and Y are 0 and 1 respectively)

Consider row–4

The product term related to row–4 is X Y (because the truth values of X and Y are 1 and 1 respectively)

Therefore the PDNF is

$$\overline{X}\,\overline{Y} \vee \overline{X}\,Y \vee X\,Y \quad \text{or} \quad \left(\overline{X} \wedge \overline{Y}\right) \vee \left(\overline{X} \wedge Y\right) \vee \left(X \wedge Y\right)$$

1.6.7 Principal Conjunctive Normal Form (PCNF)

For a given statement formula, an equivalent formula consisting of the conjunction of maxterms (product of sums) is known as Principal Conjunctive Normal Form (PCNF) (or product of sums canonical form).

1.6.8 How to Find PCNF (by using truth table or through PDNF)

Step 1: Suppose the given statement formula is F.

Step 2: Find $\overline{F}$ (the complement of F)

Step 3: Find the PDNF & $\overline{F}$ (we may use black box method or direct method)

Step 4: PCNF = $\overline{\left(\text{PDNF of } \left(\overline{F}\right)\right)}$

Example 1.47

Find the PCNF of $X \wedge \overline{Y}$.

Sol: We follow the method given in 1.6.13.

Step 1: Consider the given statement formula

$$F = \left(X \wedge \overline{Y}\right)$$

Step 2: Now we find $\overline{F}$.

$$\overline{F} = \overline{\left(X \wedge \overline{Y}\right)} = \overline{X} \vee \left(\overline{Y}\right) = \overline{X} \vee Y$$

Step 3: In this step we find the PDNF for $\overline{F} = \overline{X} \vee Y$

X	**Y**	$\overline{\mathbf{X}}$	$\overline{\mathbf{X}} \vee \overline{\mathbf{Y}}$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

Truth table for $\left(\overline{\mathbf{X}} \vee \overline{\mathbf{Y}}\right)$

As in example 1.6.11, we get that $\text{PDNF}\left(\overline{F}\right) = \left(\overline{X} \wedge \overline{Y}\right) \vee \left(\overline{X} \wedge Y\right) \vee \left(X \wedge Y\right)$

Step 4: $\text{PCNF}(F) = \overline{\text{PDNF}\left(\overline{F}\right)}$

$$= \overline{\left(\overline{X} \wedge \overline{Y}\right) \vee \left(\overline{X} \wedge Y\right) \vee \left(X \wedge Y\right)}$$

$$= \overline{\left(\overline{X} \wedge Y\right)} \wedge \overline{\left(\overline{X} \wedge Y\right)} \wedge \overline{\left(X \wedge Y\right)}$$

(by demorgan laws)

$$= \left(\overline{X} \vee \overline{Y}\right) \wedge \left(\overline{X} \vee \overline{Y}\right) \wedge \left(\overline{X} \vee \overline{Y}\right)$$

(by demorgan laws)

$$= \left(X \vee \overline{Y}\right) \wedge \left(X \vee \overline{Y}\right) \wedge \left(\overline{X} \vee \overline{Y}\right)$$

$$= \left(X \vee \overline{Y}\right) \wedge \left(\overline{X} \vee \overline{Y}\right)$$

(since $A \wedge A = A$)

Example 1.48

Construct the principal conjunctive normal form of the propositional formula.

$(\sim p \rightarrow r) \wedge (p \Leftrightarrow q)$.

[JNTUH–June 2010 Set No.1]

[JNTUH–Nov 2008 Set No.1]

Solution: Given propositional formula is $(\sim p \rightarrow r) \wedge (p \Leftrightarrow q)$ we have to find the PCNF (Principal Conjunctive Normal Formal) of $(\sim p \rightarrow r) \wedge (p \Leftrightarrow q)$.

Consider (~p→r) ∧ (p⇔q)

$\Leftrightarrow (p \vee r) \wedge [(p \to q) \wedge (q \to p)]$ [Since ~p→r⇔p∨r]

$\Leftrightarrow (p \vee r) \wedge [(\sim p \vee q) \wedge (\sim q \vee p)]$ [Since p→q⇔ ~p∨q & q→p ⇔ ~q∨p]

$\Leftrightarrow (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$

$\Leftrightarrow [p \vee r \vee (q \wedge \sim q)] \wedge [(\sim p \vee q) \vee (r \wedge \sim r)] \wedge [\sim q \vee p \vee (r \wedge \sim r)]$

$\Leftrightarrow [(p \vee r \vee q) \wedge (p \vee r \vee \sim q)] \wedge [(\sim p \vee q \vee r) \wedge (\sim p \vee q \wedge \sim r)] \wedge$

$[(\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r)]$

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

Therefore the principal conjunctive normal formal of the given propositional formula "(~p→r) ∧ (p⇔q)" is (p∨q∨r) ∧ (p∨~q∨r) ∧ (p∨~q∨~r) ∧ (~p∨q∨r) ∧ (~p∨q∨~r).

Example 1.49

Show that the principal conjunctive normal of the formula

$[p \to (q \wedge r)] \wedge [\overline{p} \to (\overline{q} \wedge \overline{r})]$ is π (1, 2, 3, 4, 5, 6)

[JNTU–Nov 2008 Set No.2]

Solution: Given formula is

$[p \to (q \wedge r)] \wedge [\overline{p} \to (\overline{q} \wedge \overline{r})]$

$\Leftrightarrow [\overline{p} \vee (q \wedge r)] \wedge [\overline{p} \to (\overline{q} \wedge \overline{r})]$ [Since p→q ⇔ ~p∨q]

$\Leftrightarrow [\overline{p} \vee (q \wedge r)] \wedge [p \vee (\overline{q} \wedge \overline{r})]$ [Since p→q ⇔ ~p∨q]

$\Leftrightarrow [(\overline{p} \vee q) \wedge (\overline{p} \vee r)] \wedge [(p \vee \overline{q}) \wedge (\overline{p} \vee \overline{r})]$ [By Demorgan laws]

$\Leftrightarrow [\overline{p} \vee q \vee (r \wedge \overline{r})] \wedge [\overline{p} \vee r \vee (q \wedge \overline{q})] \wedge [p \vee \overline{q} \vee (r \wedge \overline{r})] \wedge [p \vee \overline{r} \vee (q \wedge \overline{q})]$

$\Leftrightarrow [\overline{p} \vee q \vee r] \wedge [\overline{p} \vee q \vee \overline{r}] \wedge [\overline{p} \vee r \vee q] \wedge [\overline{p} \vee r \vee \overline{q}] \wedge [p \vee \overline{q} \vee r] \wedge [p \vee \overline{q} \vee \overline{r}]$

$\wedge [p \vee \overline{r} \vee q] \wedge [\overline{p} \vee \overline{r} \vee \overline{q}]$

$\Leftrightarrow (\overline{p} \vee q \vee r) \wedge (\overline{p} \vee q \vee \overline{r}) \wedge (\overline{p} \vee \overline{q} \vee r) \wedge (p \vee \overline{q} \vee r) \wedge (p \vee q \vee \overline{r}) \wedge (p \vee \overline{q} \vee \overline{r})$ is the principal conjunctive normal form of given statement formula.

Now by the known representation, we have

1. Can be represented as $\overline{p}\ \overline{q}\ r$ (001)

2. Can be represented as $\overline{p}\, q\, \overline{r}$ (010)
3. Can be represented as $\overline{p}\, q\, r$ (011)
4. Can be represented as $p\, \overline{q}\, \overline{r}$ (100)
5. Can be represented as $p\, \overline{q}\, r$ (101)
6. Can be represented as $p\, q\, \overline{r}$ (110)

Therefore, the principal conjunctive normal form of the given formula can be represented as π (1, 2, 3, 4, 5, 6).

It completes the solution.

Example 1.50

Obtain the canonical product of sums of the propositional formula

$$\overline{x} \wedge (\overline{y} \vee z)$$

[JNTUH, June 2010, Set-4]

Solution:

First part:

$$\overline{x} \Leftrightarrow \overline{x} \vee 0$$
$$\Leftrightarrow \overline{x} \vee (y \wedge \overline{y}) = (\overline{x} \vee y) \wedge (\overline{x} \vee \overline{y})$$
$$\Leftrightarrow [(\overline{x} \vee y) \vee 0] \wedge [(\overline{x} \vee \overline{y}) \vee 0]$$
$$\Leftrightarrow [(\overline{x} \vee y) \vee (z \wedge \overline{z})] \wedge [(\overline{x} \vee \overline{y}) \vee (z \wedge \overline{z})]$$
$$\Leftrightarrow [(\overline{x} \vee y \vee z) \wedge (\overline{x} \vee y \vee \overline{z})] \wedge [(\overline{x} \vee \overline{y} \vee z) \wedge (\overline{x} \vee \overline{y} \vee \overline{z})]$$

Second part:

$$(\overline{y} \vee z) \Leftrightarrow (\overline{y} \vee z) \vee 0$$
$$\Leftrightarrow (\overline{y} \vee z) \vee (x \wedge \overline{x})$$
$$\Leftrightarrow (\overline{y} \vee z \vee x) \wedge (\overline{y} \vee z \vee \overline{x})$$

Combining First part and Second part we get that

$$\overline{x} \wedge (\overline{y} \vee z) \Leftrightarrow [(\overline{x} \vee y \vee z) \wedge (\overline{x} \vee y \vee \overline{z})] \wedge [(\overline{x} \vee \overline{y} \vee z) \wedge (\overline{x} \vee \overline{y} \vee \overline{z})]$$
$$\wedge [(\overline{y} \vee z \vee x) \wedge (\overline{y} \vee z \vee \overline{x})]$$
$$\Leftrightarrow (\overline{x} \vee y \vee z) \wedge (\overline{x} \vee y \vee \overline{z}) \wedge (\overline{x} \vee \overline{y} \vee z) \wedge (\overline{x} \vee \overline{y} \vee \overline{z}) \wedge (\overline{y} \vee z \vee x)$$

The last expression is the required Canonical form.

Example 1.51

Obtain the canonical sum of product form for the propositional formulas:

$$\bar{x} \wedge (\bar{y} \vee z)$$

Solution: Given propositional formula is

$$\bar{x} \wedge (\bar{y} \vee z)$$

We have to obtain product of sums of given statement $\bar{x} \wedge (\bar{y} \vee z)$.

Consider $\quad \bar{x} \wedge (\bar{y} \vee z)$

$$\Leftrightarrow (\bar{x} \wedge \bar{y}) \vee (\bar{x} \wedge z)$$

$$\Leftrightarrow [\bar{x} \wedge \bar{y} \wedge (z \vee \bar{z})] \vee [\bar{x} \wedge \bar{z} \wedge (y \vee \bar{y})]$$

$$\Leftrightarrow [(\bar{x} \wedge \bar{y} \wedge z) \vee (\bar{x} \wedge \bar{y} \wedge \bar{z})] \vee [\bar{x} \wedge z \wedge y \ \vee \ \bar{x} \wedge z \wedge \bar{y}]$$

$$\Leftrightarrow (\bar{x} \wedge \bar{y} \wedge z) \vee (\bar{x} \wedge \bar{y} \wedge \bar{z}) \vee (\bar{x} \wedge y \wedge z)$$

is the required canonical sum of products form of the given propositional formula

1.7 The Theory of Inference for Statement Calculus

Logic provides the rules of inference, or principles of reasoning. The theory deal with these rules is known as inference theory. This theory is concerned with the inferring of a conclusion from the given hypothesis (or certain premises).

When we derived a conclusion from the set of given statements (or premises) by using the accepted rules, then such process of derivation is known as deduction or a formal proof. In the formal proof, at any stage, the rule of inference used in the derivation is acknowledged.

The conclusion which is arrived following the rules of inference is called as Valid Conclusion; and the argument used is called a Valid Argument.

The actual truth values of the given premises do not play any role in determining the validity of the argument.

Logic was discussed by its ancient founder Aristotle (384 BC – 322 BC) from two quite different points of view. On one hand he regarded logic as an instrument or organ for appraising the correctness or strength of the reasoning. On the other hand, he treated the principles and methods of logic as interesting and important topics of the study. The study of logic will provide the reader certain techniques for testing the validity of a given arguments. Logic provides the theoretical basis for many areas of computer science such as digital logic design, automata theory and computability, and artificial intelligence. In this chapter, we discuss the truth tables, validity of arguments using the rules of inference. Further, we study the various normal forms and logical equivalences using the rules.

1.7.1 Tautology

Suppose that A and B are two statement formulas. We say that "B logically follows from A" or "B is a valid conclusion (or consequence) of the premise A" if and only if $A \rightarrow B$ is a tautology (that is, $A \Rightarrow B$).

1.7.2 Validity using Truth Table

Let $P_1, P_2, ..., P_n$ be the variables appearing in the premises $H_1, H_2, ..., H_m$ and the conclusion C. Let all possible combinations of truth values are assigned to $P_1, P_2, ..., P_n$ and let the truth values of $H_1, H_2, ..., H_m$ and C are entered in the table. We say that C follows logically from premises $H_1, H_2, ..., H_m$ if and only if $H_1 \wedge H_2 \wedge ... H_m \Rightarrow C$. This can be checked from the truth table using the following procedure:

1. Look at the rows in which C has the value F.
2. In every such row if at least one of the values of $H_1, H_2, ..., H_m$ is F then the conclusion is valid.

Example 1.52

Show that the conclusion C: ~P follows from the premises

H_1: ~ P ∨ Q, H_2: ~ (Q ∧ ~ R) and H_3: ~ R.

Solution: Given that C: ~ P, H_1: ~ P ∨ Q, H_2: ~ (Q ∧ ~ R) and H_3: ~ R.

P	Q	R	H_1	H_2	H_3	C
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	T	T	F	T
F	T	F	T	F	T	T
F	F	T	T	T	F	T
F	F	F	T	T	T	T

The row in which C has the truth values F has the situation that at least one of H_1, H_2, H_3 has truth value F. Thus C logically follows form H_1, H_2, and H_3.

1.7.3 Rules of Inference

We now describe the process of derivation by which one demonstrates particular formula is a valid consequence of the given set of premises. The following are the three rules of inference.

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tautologically implied by any one or more of the preceding formulas in the derivation.

Rule CP: If we can derive S and R and a set of premises then we can derive $R \rightarrow S$ from the set of premises alone.

1.7.4 Some Implications

I_1	:	$p \wedge q \Rightarrow p$	(Simplification)
I_2	:	$p \wedge q \Rightarrow q$	
I_3	:	$p \Rightarrow p \vee q$	(addition)
I_4	:	$q \Rightarrow p \vee q$	
I_5	:	$\overline{p} \Rightarrow p \rightarrow q$	
I_6	:	$q \Rightarrow p \rightarrow q$	
I_7	:	$\overline{p \rightarrow q} \Rightarrow p$	
I_8	:	$\overline{p \rightarrow q} \Rightarrow \overline{q}$	
I_9	:	$p, q \Rightarrow p \wedge q$	
I_{10}	:	$\overline{p},\ p \vee q \Rightarrow q$	[disjunctive syllogism]
I_{11}	:	$p, p \rightarrow q \Rightarrow q$	[modus ponens]
I_{12}	:	$\overline{q},\ p \rightarrow q \Rightarrow \overline{p}$	[modus tollens]
I_{13}	:	$p \rightarrow q\ , q \rightarrow r \Rightarrow p \rightarrow q$	[hypothetical syllogism]
I_{14}	:	$p \vee q, p \rightarrow r, q \rightarrow r \Rightarrow r$	[dilemma]

1.7.5 Some Equivalences

E_1	:	$\overline{\overline{p}} \Leftrightarrow p$ [double negation]
E_2	:	$p \wedge q \Leftrightarrow q \wedge p$
E_3	:	$p \vee q \Leftrightarrow q \vee p$
E_4	:	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
E_5	:	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
E_6	:	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
E_7	:	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
E_8	:	$\overline{p \wedge q} \Leftrightarrow \overline{p} \vee \overline{q}$

E_9	:	$\overline{p \vee q} \Leftrightarrow \overline{p} \wedge \overline{q}$
E_{10}	:	$p \vee p \Leftrightarrow p$
E_{11}	:	$p \wedge p \Leftrightarrow p$
E_{12}	:	$r \vee (p \wedge \overline{p}) \Leftrightarrow r$
E_{13}	:	$r \wedge (p \vee \overline{p}) \Leftrightarrow r$
E_{14}	:	$r \vee (p \vee \overline{p}) \Leftrightarrow T$
E_{15}	:	$r \wedge (p \wedge \overline{p}) \Leftrightarrow F$
E_{16}	:	$p \rightarrow q \Leftrightarrow \overline{p} \vee q$
E_{17}	:	$\overline{p \rightarrow q} \Leftrightarrow p \wedge \overline{q}$
E_{18}	:	$p \rightarrow q \Leftrightarrow \overline{q} \rightarrow \overline{p}$
E_{19}	:	$p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$
E_{20}	:	$\overline{p \rightleftarrows q} \Leftrightarrow (p \rightleftarrows \overline{q})$
E_{21}	:	$(p \rightleftarrows q) \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
E_{22}	:	$p \rightleftarrows q \Leftrightarrow (p \wedge q) \vee (\overline{p} \wedge \overline{q})$

Example 1.53

Show that $r \wedge (p \vee q)$ is a valid conclusion from the premises $p \vee q$, $q \rightarrow r$, $p \rightarrow m$ and $\overline{m}$.

Solution:

{1}	(1) $p \rightarrow m$	Rule P
{2}	(2) $\overline{m}$	Rule P
{1,2}	(3) $\overline{p}$	Rule T, (1), (2) and I_{12}.
{4}	(4) $p \vee q$	Rule P
{1,2,4}	(5) q	Rule T, (3), (4) and I_{10}.
{6}	(6) $q \rightarrow r$	Rule P
{1,2,4,6}	(7) r	T, (5), (6) and I_{11}.
{1,2,4,6}	(8) $r \wedge (p \vee q)$	T, (4), (7) and I_9.

Example 1.54

Show I_{12} : $\overline{q}$, $p \rightarrow q \Rightarrow \overline{p}$

Solution:

{1}	(1) $p \rightarrow q$	Rule P
{1}	(2) $\overline{q} \rightarrow \overline{p}$	Rule T, (1) and E_{18}.

{3}	(3) $\bar{q}$	Rule P
{1,3}	(4) $\bar{p}$	Rule T, (2), (3) and I_{11}.

Example 1.55

Show that the conclusion C: ~ P follows from the premises

H_1: ~ P ∨ Q, H_2: ~ (Q ∧ ~ R) and H_3: ~ R.

Solution: We get

	(1) ~ R	Rule P (assumed premise)
	(2) ~ (Q ∧ ~ R)	Rule P
{2}	(3) ~ Q ∨ R	Rule T
{3}	(4) R ∧ ~ Q	Rule T
{4}	(5) ~ R → ~ Q	Rule T
{1, 5}	(6) ~ Q	Rule T
	(7) ~ P ∨ Q	Rule P
{7}	(8) ~ Q → ~ P	Rule T
{6, 8}	(9) ~ P	Rule T

So C logically follows from H_1, H_2 and H_3.

Example 1.56

Show that S ∨ R is tautologically implied by (P ∨ Q) ∧ (P → R) ∧ (Q → S).

Solution: Note that P ∨ Q, P → R, Q → S are three premises.

	(1) P ∨ Q	Rule P
{1}	(2) ~ P → Q	Rule T
	(3) Q → S	Rule P
{2, 3}	(4) ~ P → S	Rule T
	(5) ~ S → P	Rule T (as P → Q ⇔ ~ Q → ~ P)
	(6) P → R	Rule P
{5, 6}	(7) ~ S → R	Rule T
{7}	(8) S ∨ R	Rule T

Example 1.57

Show that R → S can be derived from the premises P → (Q → S), ~ R ∨ P and Q.

Solution: We get

	(1) R	Rule P
	(2) ~ R ∨ P	Rule P
{2}	(3) R → S	Rule T

{1, 3}	(4) P	Rule T
	(5) $P \to (Q \to S)$	Rule P
{4, 5}	(6) $Q \to S$	Rule T
	(7) Q	Rule P
{7, 6}	(8) S	Rule T
	(9) $R \to S$	Rule CP

Example 1.58

Prove or disprove the conclusion given below from the following axioms.

"If Socrates is a man, Socrates is mortal. Socrates is a man." Therefore Socrates is mortal.

[JNTUH, Nov 2010, Set No.3]

Solution: The argument is valid because it follows the pattern of Modus ponens.

Consider the argument

p: Socrates is a man.

q: Socrates is mortal.

$p \to q$: If Socrates is a man, then Socrates is mortal.

The modus ponens is

$$\begin{array}{l} p \to q \\ \underline{p \quad\quad} \\ \therefore q \end{array}$$

Therefore, Socrates is mortal is true.

1.8 Consistency of Premises and Indirect Method of Proof

A set of m formulas H_1, H_2, ...H_m is said to be consistent if their conjunction ($H_1 \wedge$, $H_2 \wedge ... \wedge H_m$) has truth value "T" for some assignment of the truth values to the atomic variables appearing in the statement formulas H_1, H_2, ...H_m.

If $H_1 \wedge$, $H_2 \wedge ... \wedge H_m$ is false for every assignment of the truth values of the automic variables appearing in the statement formulas H_1, H_2, ...H_m then we say that H_1, H_2, ...H_m are inconsistent. In other words, a set of formulas H_1, H_2, ...H_m are inconsistent if their conjunction ($H_1 \wedge$, $H_2 \wedge ... \wedge H_m$) implies a contradiction, that is,

$$H_1 \wedge, H_2 \wedge ... \wedge H_m \Rightarrow R \wedge \overline{R}$$

where R is any statement formula. Note that $R \wedge \overline{R}$ is a contradiction for any formula R].

This concept will be used in a procedure called "proof by contradiction" (or indirect method of proof).

1.8.1 Indirect Method of Proof

In order to show that a conclusion C follows logically from the premises $H_1, H_2, ..., H_m$ we assume that C is FALSE and consider ~C as an additional premise. If $H_1 \wedge H_2 \wedge ... \wedge H_m \wedge \sim C$ is a contradiction, then C follows logically from $H_1, H_2, ..., H_m$.

Example 1.59

Show that $\sim (P \wedge Q)$ follows from $\sim P \wedge \sim Q$.

Solution: Assume $\sim (\sim (P \wedge Q))$ as an additional premise. Then

	(1) $\sim (\sim (P \wedge Q))$	Rule P
{1}	(2) $P \wedge Q$	Rule T
	(3) P	Rule T
	(4) $\sim P \wedge \sim Q$	Rule P
{4}	(5) $\sim P$	Rule T
{3, 5}	(6) $P \wedge \sim P$	Rule T

Therefore $P \wedge \sim P$ is a contradiction. Hence by the indirect method of proof $\sim(P \wedge Q)$ follows from $\sim P \wedge \sim Q$.

Example 1.60

"If there was a party then catching the train was difficult. If they arrived on time then catching the train was not difficult. They arrived on time. Therefore there was no party." Show that the statement constitutes a valid argument.

Solution:

Let p: There was a party

q: Catching the train was difficult.

r: They arrived on time.

We have to prove $\overline{P}$ follows from the premises $p \rightarrow q$, $r \rightarrow \overline{q}$ and r.

	(1) r	Rule P
	(2) $r \rightarrow \overline{q}$	Rule P
{1, 2}	(3) $\overline{q}$	Rule T
	(4) $p \rightarrow q$	Rule P
{4}	(5) $\overline{q} \rightarrow \overline{p}$	Rule T
{3, 5}	(6) $\overline{p}$	Rule T

Example 1.61

How does an indirect proof technique differ from a direct proof.

[JNTUH, June 2010, Set No.2]

[JNTUH, Nov 2008, Set No.3]

Solution: We know that a set of formulas $H_1, H_2, ..., H_m$ is said to be consistent if their conjunction has the truth value T for some assignment of the truth value to the atomic variables appearing in $H_1, H_2, ..., H_m$.

If for every assignment of the truth values to the atomic variables, their conjunction $H_1 \wedge H_2 \wedge ... \wedge H_m$ is identically false, then the formulas $H_1, H_2, ..., H_m$ are called inconsistent.

In other words, a set of formulas $H_1, H_2, ..., H_m$ is inconsistent if their conjunction $H_1 \wedge H_2 \wedge ... \wedge H_m$ implies a contradiction, that is,

$$H_1 \wedge H_2 \wedge ... \wedge H_m \Rightarrow R \wedge \urcorner R$$

where R is any formula.

Note that $R \wedge \urcorner R$ is a contradiction, for any formula R.

In "Direct Method of Proof" we prove directly the conclusion by using the set of premises and inference rules. But the notion of inconsistency is used in a procedure called "proof by contradiction" (or "reduction and absurdum") or "indirect method of proof".

In this indirect method of proof, to show that a conclusion 'C' follows logically from the premises $H_1, H_2, ..., H_m$, we assume that 'C' is false and consider $\urcorner C$ as an additional premise. If the new set of premises $H_1, H_2, ..., H_m, \urcorner C$ are inconsistent, then the assumption that "$\urcorner C$ is true" does not hold simultaneously with $H_1 \wedge H_2 \wedge ... \wedge H_m$ being true.

There C is true whenever $H_1 \wedge H_2 \wedge ... \wedge H_m$ is true. This concept is used in Indirect Method of Proof.

Proof by Indirect Method is sometimes convenient. However it can always be eliminated and replaced by Conditional Proof (CP).

Consider the following.

$$P \rightarrow (Q \wedge \urcorner Q) \Rightarrow \urcorner P \qquad .....(1.14)$$

In the proof by indirect method, we show that

$$H_1, H_2, ..., H_m \Rightarrow C$$

by proving that

$$H_1, H_2, ..., H_m, \urcorner C \Rightarrow R \wedge \urcorner R \qquad .....(1.15)$$

for same formula R.

Now (ii) can be written (by using rule C.P) as follows

$H_1, H_2, ..., H_m, \Rightarrow \neg C$

$\rightarrow (R \wedge \neg R)$ .....(1.16)

From (iii) and (i) and "$\neg\neg P \Leftrightarrow P$" we get that

$H_1, H_2, ..., H_m, \Rightarrow C$

which is the required derivation. Hence in some cases, the indirect method of proof is more convenient.

Example 1.62

Show that the following set of premises are inconsistent using indirect method of proof.

$p \rightarrow q, q \rightarrow r, \sim (p \wedge r), p \vee r \Rightarrow r$

[JNTUH, Nov 2008, Set No.2]

Solution: We wish to state that the given set of premises is not inconsistent.

Reason: when (p, q, r) = (F, T, T), that is, the truth values of p, q, r are equal to F, T, T respectively, then the truth values of $p \rightarrow q$, $q \rightarrow r$, $\sim (p \wedge r)$ and $p \vee r \rightarrow r$ are all equal to T.

Hence $p \rightarrow q$, $q \rightarrow r$, $\sim (p \wedge r)$, $p \vee r \rightarrow r$ will not form a set of inconsistent formula.

Exercises

Statements and Notations

1. Write down the negation of the statement "No real number is greater than its square."

Ans: At least one real number is greater than its square.

2. Write down the following statement in symbolic form, and find its negation:

   "If all triangles are right-angled, then no triangle is equiangular"

Ans: $\{\forall x \in T, p(x)\} \rightarrow \{\forall x \in T, \sim q(x)\}$

The negation is

$\{\forall x \in T, p(x)\} \wedge \{\exists x \in T, q(x)\}$

3. Prove that for any three statements p, q, r

***Ans*:** $[(p \vee q) \rightarrow r] \Leftrightarrow \{(p \rightarrow r) \wedge (q \rightarrow r)\}$

4. Let p and q be primitive statements for which the implication $p \rightarrow q$ is false. Determine the truth value of (i) $p \wedge q$ (ii) $(\sim p) \vee q$ (iii) $q \rightarrow p$ (iv) $(\sim q) \rightarrow (\sim p)$

***Ans*:** (i) 0 (ii) 0 (iii) 1 (iv) 0

Connectives and Truth Tables

1. Using the following statements

   p: Raju is rich

   q: Raju is happy

   Write the following statements in symbolic form

   (i) Raju is rich or unhappy

   (ii) Raju is neither rich nor happy

   (iii) Raju is poor, but happy

***Ans*:**

(i) $(p \vee \sim q)$

(ii) $(\sim p \wedge \sim q)$

(iii) $(\sim p \wedge q)$

2. Construct the truth table for the following

   (i) $[(p \wedge q) \vee (\sim r)] \rightleftarrows q$

   (ii) $(p \wedge q) \vee (\sim p \wedge \sim r)$

   (iii) $[(p \vee q) \wedge (\sim p \vee r)] \wedge (q \vee r)$

***Ans*:**

p	q	r	~r	$p \wedge q$	$[(p \wedge q) \vee \sim r]$	$[(p \wedge q) \vee \sim r] \rightleftarrows q$
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	F	T	F	F	F	T
T	F	F	T	F	T	F
F	T	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	F	F	F	F
F	F	F	T	F	T	T

3. Given that p is true and q is false, find the truth values of the following
   (i) $(\sim p \vee q) \wedge (\sim q \rightleftarrows p)$
   (ii) $(p \vee q) \rightarrow (p \wedge q)$
   (iii) $(p \rightarrow \sim q) \wedge (q \rightarrow \sim p)$

***Ans*:**

(i) False
(ii) False
(iii) True

Tautology and Contradiction

1. Prove the following are tautology
   (i) $\left[\left[p \rightarrow (q \vee r)\right] \wedge \sim q\right] \rightarrow (p \rightarrow r)$
   (ii) $\left[(p \wedge \sim q) \rightarrow r\right] \rightarrow \left[p \rightarrow (q \vee r)\right]$
   (iii) $\sim (p \vee q) \vee [\sim p \wedge q] \vee p$
2. Prove the following contradiction
   (i) $(p \vee q) \wedge (p \rightleftarrows q)$ (ii) $p \wedge \sim p$
3. Show that the statement $(p \wedge q) \Rightarrow p$ is a tautology

***Ans*:**

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Truth table for $(p \wedge q) \Rightarrow p$ which is a tautology

Equivalence of Statements or Formulas

1. Prove the following logical equivalence
   (i) $\left[p \rightarrow (q \rightarrow r)\right] \Leftrightarrow \left[p \rightarrow (\sim q \vee r)\right] \Leftrightarrow \left[(p \wedge q) \rightarrow r\right] \Leftrightarrow (p \rightarrow r) \vee (q \rightarrow r)$
   (ii) $\left[p \rightarrow (q \vee r)\right] \Leftrightarrow \left[(p \rightarrow q) \vee (p \rightarrow r)\right] \Rightarrow \left[\sim r \rightarrow (p \rightarrow q)\right] \Rightarrow \left[(p \wedge \sim q) \rightarrow r\right]$
   (iii) $\left[(p \rightarrow q) \wedge (r \rightarrow q)\right] \Leftrightarrow \left[(p \vee r) \rightarrow q\right]$

2. Show the following equivalence
   (i) $(\sim p \wedge \sim q) \Leftrightarrow \sim (p \vee q)$
   (ii) $\sim (p \rightleftarrows q) \Leftrightarrow (p \wedge \sim q) \vee (\sim p \wedge q)$
3. Prove that

$$\left[(\sim p \vee q) \wedge (p \wedge (p \wedge q))\right] \Leftrightarrow p \wedge q$$

Hence deduce that

$$\left[(\sim p \wedge q) \vee (p \vee (p \vee q))\right] \Leftrightarrow p \vee q$$

Duality Law and Tautological Implication

1. Prove the following implications
   (i) $\left[p \to (q \to r)\right] \Rightarrow (p \to q) \to (p \to r)$
   (ii) $p \Rightarrow (q \to p)$
2. Prove that $\left[(\sim p \vee q) \wedge p \wedge (p \wedge q)\right] \Leftrightarrow p \wedge q$ by duality principle
3. Verify the principle duality for the logical equivalence

$$(p \vee q) \wedge (\sim p \wedge (\sim p \wedge q)) \Leftrightarrow (\sim p \wedge q)$$

Normal Forms

1. Obtain the principal disjunctive Normal Form for
   (i) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$
   (ii) $p \to \left[(p \to q) \wedge \sim (\sim q \vee \sim p)\right]$

Ans: $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

2. Find the DNF and CNF of the following formulas
   (i) $(\sim p \to r) \wedge (q \rightleftarrows p)$
   (ii) $p \to \left[p \wedge (q \to p)\right]$

Ans: ???

3. Find the principal disjunctive normal form of the following
   (i) $\sim (p \wedge q)$
   (ii) $p \rightleftarrows q$

Ans:

(i) $(p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

(ii) $(p \wedge q) \vee (\sim p \wedge \sim q)$

4. Find the principal conjunctive normal Form of the following

(i) $(\sim p) \wedge q$

(ii) $\sim (p \vee q)$

Ans:

(i) $(\sim p \vee q) \wedge (\sim p \vee q) \wedge (p \vee q)$

(ii) $(p \vee \sim q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)$

The Theory of Inference for Statement Calculus

1. Prove the argument $(p \rightarrow q) \wedge (r \rightarrow s), (p \vee r) \wedge (q \vee r) / -q \vee s$ is valid without using truth table
2. Show that $\sim (p \wedge q)$ follows from $\sim p \wedge \sim q$ without using truth table
3. Prove the following using the rule CP if necessary

(i) $p \rightarrow q \Rightarrow p \rightarrow (p \wedge q)$

(ii) $p, \; p \rightarrow [q \rightarrow (r \wedge s)] \Rightarrow q \rightarrow s$

Consistency of Premises and Indirect Method of Proof

1. Prove that "the square of an even integer is an even integer" by the method of contradiction.
2. Provide an indirect proof of the following statement For all positive real numbers x and y, if the product xy exceeds 25, then x > 5 or y > 5.
3. Show that the following sets of premises are inconsistent.

$$p \rightarrow (q \rightarrow r), s \rightarrow (q \wedge \sim r), p \wedge s$$

4. Show the following (use indirect method if need)

(i) $(r \rightarrow \sim q), r \vee s, s \rightarrow \sim q, p \rightarrow q \Rightarrow \sim p$

(ii) $\sim (p \rightarrow q) \rightarrow \sim (r \vee s), [(q \rightarrow p) \vee \sim r], R \Rightarrow P \rightleftarrows q$