Chapter

1

Static Electric Fields

1.1 Introduction

Electrostatic in the sense static or rest or time in-varying electric fields. Electrostatic field can be obtained by the distribution of static charges.

The two fundamental laws which describe electrostatic fields are Coulomb's law and Gauss's law:

They are independent laws. i.e., one law does not depend on the other law.

Coulomb's law can be used to find electric field when the charge distribution is of any type, but it is easy to use Gauss's law to find electric field when the charge distribution is symmetrical.

1.2 Coulomb's Law

This law is formulated in the year 1785 by Coulomb. It deals with the force a point charge exerts on another point charge; generally a charge can be expressed in terms of coulombs.

$$1 \text{ coulomb} = 6 \times 10^{18} \text{ electrons}$$

$$1 \text{ electron charge} = -1.6 \times 10^{-19} \text{ Coulombs}$$

Coulomb's law states that the force between two point charges Q_1 and Q_2 is along the line joining between them, directly proportional to the product of two point charges, and inversely proportional to the square of the distance between them

$$\therefore \qquad F = \frac{KQ_1Q_2}{R^2}$$

where K is proportional constant

In SI, a unit for Q_1 and Q_2 is coulombs(C), for R meters(m) and for F newtons(N).

$$K = \frac{1}{4\pi \in_0}$$

where $\in_0$ = permittivity of free space (or) vacuum

$= 8.854 \times 10^{-12}$ farads/meter

$= \frac{10^{-9}}{36\pi}$ farads/m

$$K = \frac{36\pi}{4\pi \times 10^{-9}} = 9 \times 10^9 \text{ m/farads}$$

$$F = \frac{Q_1Q_2}{4\pi \in_0 R^2} \qquad \text{.....(1.2.1)}$$

Assume that the point charges Q_1 and Q_2 are located at (x_1, y_1, z_1) and (x_2, y_2, z_2) with the position vectors $\overline{r_1}$ and $\overline{r_2}$ respectively. Let the force on Q_2 due to Q_1 be $\overline{F_{12}}$ which can be written as

$$\overline{F}_{12} = \frac{Q_1Q_2}{4\pi \in_0 R^2}\overline{a}_{R12} \qquad \text{.....(1.2.2)}$$

where $\overline{a}_{R12}$ is unit vector along the vector $\overline{R}_{12}$. Graphical representation of the vectors in rectangular coordinate system is shown in Fig.1.1

Where $\overline{a}_x$ is the unit vector along X-axis and $\overline{a}_y$ is the unit vector along Y-axis and $\overline{a}_z$ is the unit vector along Z-axis.

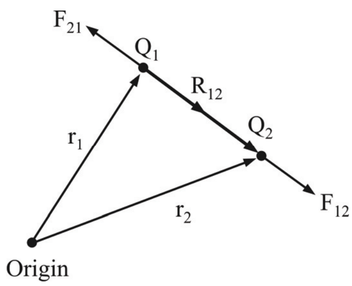


Fig. 1.1 Graphical representation of the vectors

From Fig.1.1, we can write $\overline{r}_1 + \overline{R}_{12} = \overline{r}_2$

i.e., $\overline{R}_{12} = \overline{r}_2 - \overline{r}_1$

where $\overline{r}_1 = x_1\overline{a}_x + y_1\overline{a}_y + z_1\overline{a}_z$

$\overline{r}_2 = x_2\overline{a}_x + y_2\overline{a}_y + z_2\overline{a}_z$

Now $$\overline{F}_{12} = \frac{Q_1Q_2}{4\pi \in_0 R^2}\frac{\overline{R}_{12}}{\left|\overline{R}_{12}\right|}$$

$$\because \quad \overline{a}_{R12} = \frac{\overline{R}_{12}}{\left|\overline{R}_{12}\right|}$$

$$= \frac{Q_1Q_2}{4\pi \in_0 R^2}\frac{\overline{R}_{12}}{R} \qquad \because \quad \left|R_{12}\right| = R$$

$$= \frac{Q_1Q_2}{4\pi \in_0}\frac{\overline{R}_{12}}{R^3}$$

$$= \frac{Q_1Q_2}{4\pi \in_0}\frac{\overline{r}_2 - \overline{r}_1}{\left|\overline{r}_2 - \overline{r}_1\right|^3} \qquad \text{.....(1.2.3)}$$

and force on Q_1 due to Q_2 is $\overline{F}_{21} = -\overline{F}_{12}$

If we have more than two point charges i.e., Q_1, Q_2,… Q_N with the position vectors $\overline{r}_1, \overline{r}_2, ...\overline{r}_N$ respectively, then the force on a point charge Q, whose position vector is $\overline{r}$, can be written as

$$F = \frac{QQ_1}{4\pi \in_0}\frac{\overline{r} - \overline{r}_1}{\left|\overline{r} - \overline{r}_1\right|^3} + \frac{QQ_2}{4\pi \in_0}\frac{\overline{r} - \overline{r}_2}{\left|\overline{r} - \overline{r}_2\right|^3} + ... + \frac{QQ_N}{4\pi \in_0}\frac{\overline{r} - \overline{r}_N}{\left|\overline{r} - \overline{r}_N\right|^3}$$

$$= \frac{Q}{4\pi \in_0}\sum_{K=1}^{N} Q_K \frac{\overline{r} - \overline{r}_K}{\left|\overline{r} - \overline{r}_K\right|^3} \qquad \text{.....(1.2.4)}$$

1.3 Electric Field Intensity

Electric field intensity is defined as force per unit charge in an electric field. The other name of electric field intensity is electric field strength and it is denoted by $\overline{E}$.

$$\therefore \qquad \overline{E} = \frac{\overline{F}}{Q} \text{ N/C or Volts/meter}$$

i.e.,
$$\overline{E} = \frac{Q\,Q}{Q4\pi \in_0 R^2} = \frac{Q}{4\pi \in_0 R^2} \qquad \text{.....(1.3.1)}$$

Consider a point charge Q with position vector $\overline{r}$, then the electric field intensity $\overline{E}$ at some point with position vector $\overline{r}_1$ due to point charge Q is

$$\overline{E} = \frac{Q}{4\pi \in_0 R^2}\overline{a}_R \qquad \text{.....(1.3.2)}$$

where $\overline{a}_R$ is the unit vector along $\overline{R}$. Graphical representation of vector is shown in Fig.1.2

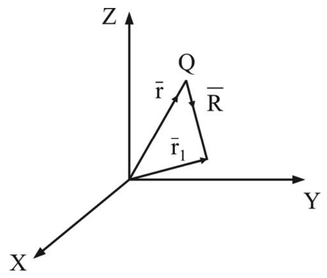


Fig. 1.2 Graphical representation

From Fig. 1.2, $\overline{R} = \overline{r}_1 - \overline{r}$

$$\overline{E} = \frac{Q}{4\pi \in_0 R^2}\frac{\overline{R}}{|\overline{R}|}$$

$$= \frac{Q}{4\pi \in_0 R^2}\frac{\overline{R}}{R}$$

$$= \frac{Q}{4\pi \in_0}\frac{\overline{R}}{R^3} = \frac{Q}{4\pi \in_0}\frac{\overline{r}_1 - \overline{r}}{|\overline{r}_1 - \overline{r}|^3}$$

If we have more than one point charge i.e., Q_1, Q_2,… Q_N with the position vectors $\overline{r}_1, \overline{r}_2, \ldots\overline{r}_N$ respectively. Then the electric field intensity $\overline{E}$ at some point with position vector $\overline{r}$ can be written as

$$\overline{E} = \frac{Q_1}{4\pi \in_0}\frac{\overline{r} - \overline{r}_1}{|\overline{r} - \overline{r}_1|^3} + \frac{Q_2}{4\pi \in_0}\frac{\overline{r} - \overline{r}_2}{|\overline{r} - \overline{r}_2|^3} + \ldots + \frac{Q_N}{4\pi \in_0}\frac{\overline{r} - \overline{r}_N}{|\overline{r} - \overline{r}_N|^3}$$

$$= \frac{1}{4\pi \in_0}\sum_{K=1}^{N} Q_K \frac{\overline{r} - \overline{r}_K}{|\overline{r} - \overline{r}_K|^3} \qquad \text{.....(1.3.3)}$$

Problem 1.1

Point charges 1 mC and –2 mC are located at (3, 2, –1) and (–1, –1, 4) respectively. Calculate the electric force on a 10 nC charge located at (0, 3, 1) and the electric field intensity at that point.

Solution

We know

$$\overline{F} = \frac{Q}{4\pi \in_0}\sum_{K=1}^{2} Q_K \frac{\overline{r} - \overline{r}_K}{|\overline{r} - \overline{r}_K|^3}$$

$$= \frac{10\times10^{-9}}{4\pi \in_0}\left[1\times10^{-3}\frac{\left(-3\overline{a}_x + \overline{a}_y + 2\overline{a}_z\right)}{\left(\sqrt{9+1+4}\right)^3} - 2\times10^{-3}\frac{\left(\overline{a}_x + 4\overline{a}_y - 3\overline{a}_z\right)}{\left(\sqrt{1+16+9}\right)^3}\right]$$

$\because \quad \in_0 = 8.854\times10^{-12}$ and $\pi = 3.14$

$$= 90\left[\frac{\left(-3\overline{a}_x + \overline{a}_y + 2\overline{a}_z\right)\times10^{-3}}{52.38} - 10^{-3}\frac{\left(2\overline{a}_x + 8\overline{a}_y - 6\overline{a}_z\right)}{132.57}\right]$$

$$= 90\times10^{-3}\left[\overline{a}_x\left(\frac{-3}{52.38} - \frac{2}{132.57}\right) + \overline{a}_y\left(\frac{1}{52.38} - \frac{8}{132.57}\right) + \overline{a}_z\left(\frac{2}{52.38} + \frac{6}{132.57}\right)\right]$$

$$= 90\times10^{-3}\left[-0.0723\overline{a}_x - 0.0413\overline{a}_y + 0.0834\overline{a}_z\right]$$

$$= -0.0065\,\overline{a}_x - 0.0037\,\overline{a}_y + 0.0075\,\overline{a}_z \text{ N.}$$

Also we know $\overline{E} = \dfrac{\overline{F}}{Q}$

$$= -\frac{0.0065}{10\times10^{-9}}\overline{a}_x - \frac{0.037}{10\times10^{-9}}\overline{a}_y + \frac{0.0075}{10\times10^{-9}}\overline{a}_z$$

$$= -650\overline{a}_x - 370\overline{a}_y + 750\overline{a}_z \text{ kV/m.}$$

Problem 1.2

Point charges 5 nC and –2 nC are located at $2\overline{a}_x + 4a_z$ and $-3\overline{a}_x + 5\overline{a}_z$ respectively. (a) Determine the force on a 1 nC point charge located at $\overline{a}_x - 3\overline{a}_y + 7\overline{a}_z$. (b) Find the electric field $\overline{E}$ *at* $\overline{a}_x - 3\overline{a}_y + 7\overline{a}_z$.

Solution

(a) We know

$$\overline{F} = \frac{Q}{4\pi\in_0}\sum_{K=1}^{2} Q_K \frac{\overline{r} - \overline{r}_K}{\left|\overline{r} - \overline{r}_K\right|^3}$$

$$= 10^{-9}\times9\times10^{9}\times10^{-9}\left[5\frac{\left(-\overline{a}_x - 3\overline{a}_y + 3\overline{a}_z\right)}{\left(\sqrt{1+9+9}\right)^3} - \frac{2\left(4\overline{a}_x - 3\overline{a}_y + 2\overline{a}_z\right)}{\left(\sqrt{16+9+4}\right)^3}\right]$$

$$= 9\times10^{-9}\left[\overline{a}_x\left(\frac{-5}{82.81} - \frac{8}{156.169}\right) + \overline{a}_y\left(\frac{-15}{82.81} + \frac{6}{156.169}\right) + \overline{a}_z\left(\frac{15}{82.81} - \frac{4}{156.169}\right)\right]$$

$$= 9\times10^{-9}\left[\overline{a}_x(-0.112) + \overline{a}_y(-0.143) + \overline{a}_z(0.155)\right]$$

$$= -1.008\overline{a}_x - 1.287\overline{a}_y + 1.395\overline{a}_z \text{ nN}$$

(b) $\bar{E} = \frac{\bar{F}}{Q}$, here Q = 1 nC

$$\therefore \quad \bar{E} = -1.008\bar{a}_x - 1.287\bar{a}_y + 1.395\bar{a}_z \ \text{V/m}$$

Problem 1.3

Point charges Q_1 and Q_2 are respectively located at (4, 0, –3) and (2, 0, 1). If Q_2 = 4 nC, Find Q_1 such that (a) The $\bar{E}$ at (5, 0, 6) has no Z-component. (b) The force on a test charge at (5, 0, 6) has no X-component.

Solution

We have $\bar{F} = \frac{Q}{4\pi \in_0} \sum_{K=1}^{2} Q_K \frac{\bar{r} - \bar{r}_K}{|\bar{r} - \bar{r}_K|^3}$

(a) $\bar{E} = \frac{\bar{F}}{Q} = \frac{1}{4\pi \in_0}\left[\frac{Q_1 |(5,0,6)-(4,0,-3)|}{\left(\sqrt{1+81}\right)^3} + \frac{4\times 10^{-9} |(5,0,6)-(2,0,1)|}{\left(\sqrt{9+25}\right)^3}\right]$

Given $\bar{E}$ has no Z – component, considering only Z components on both sides

$$0 = \frac{1}{4\pi \in_0}\left[\frac{Q_1 \times 9}{\left(\sqrt{82}\right)^3} + \frac{4\times 10^{-9}\times 5}{\left(\sqrt{34}\right)^3}\right]$$

$$\frac{Q_1 \times 9}{\left(\sqrt{82}\right)^3} = -\frac{4\times 10^{-9}\times 5}{\left(\sqrt{34}\right)^3}$$

$$Q_1 = -\frac{20}{9}\left(\sqrt{\frac{41}{17}}\right)^3 nC = -8.3\,\text{nC}$$

(b) Given the force on test charge has no X-component

$$0 = \frac{Q}{4\pi \in_0}\left[\frac{Q_1}{\left(\sqrt{82}\right)^3} + \frac{4\times 10^{-9}\times 3}{\left(\sqrt{34}\right)^3}\right]$$

$$= \frac{Q_1}{\left(\sqrt{82}\right)^3} = -\frac{4\times 10^{-9}\times 3}{\left(\sqrt{34}\right)^3}$$

$$Q_1 = -12\left(\sqrt{\frac{41}{17}}\right)^3 \text{nC} = -44.95\,\text{nC}$$

Problem 1.4

Two point charges of equal mass 'm', charge 'Q' are suspended at a common point by two threads of negligible mass and length 'l'. Show that at equilibrium the inclination angle 'α' of each thread to the vertical is given by $Q^2 = 16\ \pi\in_0 mgl^2 \sin^2\alpha \tan\alpha$, (or) $\dfrac{\tan^3\alpha}{1+\tan^2\alpha} = \dfrac{Q^2}{16\pi\in_0 mgl^2}$,

if 'α' is very small

Show that $\alpha = \sqrt[3]{\dfrac{Q^2}{16\pi\in_0 mgl^2}}$

Solution:

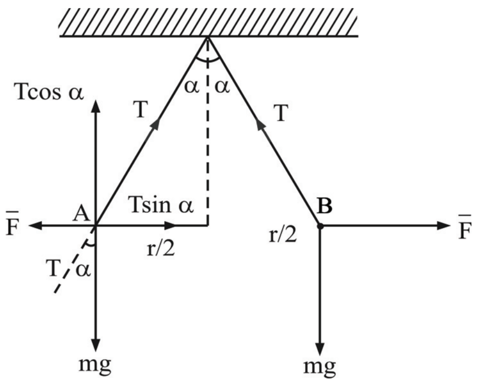


Fig. 1.3 Suspended charge particles

When two charges are suspended from a common point with threads of length 'l', we can represent graphically as sown in Fig.1.3, where T is the tension in thread 'mg' is the weight of charge towards ground due to gravitational force and $\overline{F}$ is force on charge at 'A'(B) due to charge at 'B'(A). $T\cos\alpha$ is the vertical component of 'T' which is upwards and $T\sin\alpha$ is the horizontal component of 'T' which is opposite to $\overline{F}$. To form equilibrium either at 'A' or 'B'

$$T\cos\alpha = mg \qquad \text{.....(1.3.4)}$$

$$T\sin\alpha = \overline{F} \qquad \text{.....(1.3.5)}$$

$$\frac{(1.3.4)}{(1.3.5)} = \frac{T\sin\alpha}{T\cos\alpha} = \frac{\overline{F}}{mg}$$

$\Rightarrow \qquad \tan\alpha = \frac{\bar{F}}{mg}$

where $\qquad \bar{F} = \frac{Q^2}{4\pi \in_0 r^2}$

From Fig.1.3 $\qquad \sin\alpha = \frac{r/2}{l}$

$$\Rightarrow r = 2l\sin\alpha$$

$$\tan\alpha = \frac{Q^2}{4mg\pi \in_0 r^2}$$

$$= \frac{Q^2}{4mg\pi \in_0 4l^2 \sin^2\alpha}$$

$$\tan\alpha = \frac{Q^2}{16mgl^2\pi \in_0 \sin^2\alpha}$$

$$\sin^2\alpha \tan\alpha = \frac{Q^2}{16mgl^2\pi \in_0} \qquad \text{.....(1.3.6)}$$

$$\Rightarrow \quad Q^2 = 16\pi \in_0 mgl^2 \sin^2\alpha \tan\alpha \qquad \text{.....(1.3.7)}$$

From (1.3.6)

$$\cos^2\alpha \frac{\sin^2\alpha}{\cos^2\alpha} \tan\alpha = \frac{Q^2}{16\pi \in_0 mgl^2}$$

$$\frac{\tan^3\alpha}{\sec^2\alpha} = \frac{Q^2}{16\pi \in_0 mgl^2}$$

$$\frac{\tan^3\alpha}{1+\tan^2\alpha} = \frac{Q^2}{16\pi \in_0 mgl^2}$$

If α is very small, sin α = tan α = α

From (1.3.4) $Q^2 = 16\pi\in_0 mg\, l^2\, \alpha^3$

$$\alpha^3 = \frac{Q^2}{16\pi \in_0 mg\, l^2}$$

$$\alpha = \sqrt[3]{\frac{Q^2}{16\pi \in_0 mg\, l^2}}$$

Problem 1.5

Two small identical conducting spheres have charges of 2×10^{-9} and -0.5×10^{-9} C respectively. (a) When they are placed 4 cm apart what is the force between them? (b) If they are brought into contact and then separated by 4 cm. What is the force between them?

Solution

(a) We know

$$\overline{F} = \frac{Q_1 Q_2}{4\pi \in_0 R^2}$$

$$\because \frac{1}{4\pi \in_0} = 9 \times 10^9$$

$$\overline{F} = \frac{-2 \times 10^{-9} \times 0.5 \times 10^{-9} \times 9 \times 10^9}{16 \times 10^{-4}}$$

$$= -5.625 \ \mu\text{N}$$

(b) When they are brought into contact, charges will be added and again when they are separated charge will be distributed equally

$Q_1 = 0.758 \times 10^{-9}$ C

$Q_2 = 0.75 \times 10^{-9}$ C

$\overline{F} = 3.164$ μN

Problem 1.6

If the charges in the above problem are separated with the same distance in a kerosene ($\in_r = 2$), then find (a) and (b) as in the previous problem.

Solution

(a) $\overline{F}_k = \frac{-5.625}{2}$ μN

$= -2.8125$ μN

(b) $\overline{F}_k = \frac{3.164}{2} = 1.582$ μN

Problem 1.7

Three equal +Ve charges of 4×10^{-9} C each are located at 3 corners of a square, side 20 cm. Determine the magnitude and direction of the electric field at the vacant corner point of the square.

Solution

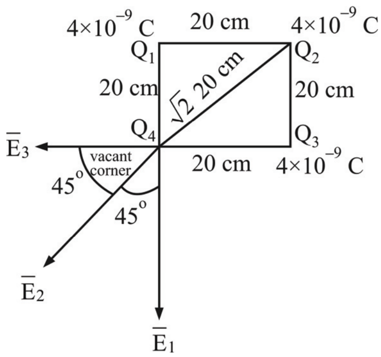


Fig. 1.4

$\overline{E}_1$ = Electric field intensity at Q_4 due to Q_1

$$= \frac{Q_1}{4\pi \in_0 R^2}$$

$$= 900 \text{ V/m}$$

$\overline{E}_2 = 450$ V/m

$\overline{E}_3 = 900$ V/m

The electric field intensity at vacant point is

$$\overline{E} = \overline{E}_2 + \overline{E}_1 \cos 45^\circ + \overline{E}_3 \cos 45^\circ$$

$$= 450 + \frac{900}{\sqrt{2}} + \frac{900}{\sqrt{2}}$$

$$= 450 + 900\sqrt{2}$$

$$= 1722.792206 \text{ V/m}$$

1.4 Coordinate Systems

The most widely used coordinate systems are Cartesian or rectangular co-ordinate system, Circular or cylindrical co-ordinate system, and Spherical co-ordinate system.

1.4.1 Cartesian Co-ordinate System

In this system the co-ordinates are X, Y, Z in which three are mutually perpendicular to each other. This system is shown in Fig. 1.5, where $\bar{a}_x, \bar{a}_y$ & $\bar{a}_z$ are unit vectors along X, Y and Z respectively. In Cartesian co-ordinate system the dot product of any unit vector with itself gives '1'.

i.e., $\bar{a}_x \cdot \bar{a}_x = 1$ $\quad = 1$ $\quad \bar{a}_z \cdot \bar{a}_z = 1$

and the dot product of one unit vector with the other one gives '0'.

i.e., $\bar{a}_x \cdot \bar{a}_y = 0$ $\quad \bar{a}_y \cdot \bar{a}_z = 0$ $\quad \bar{a}_z \cdot \bar{a}_x = 0$

The cross product of one unit vector with the other unit vector, which is next to the first one in anticlockwise direction, results the last unit vector in anticlockwise direction.

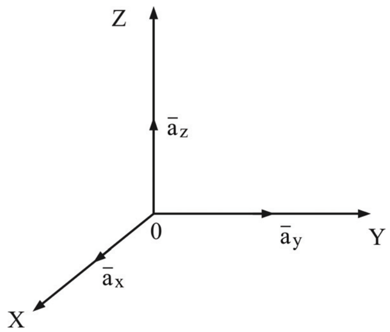


Fig. 1.5 Cartesian co-ordinate system

i.e., $\bar{a}_x \times \bar{a}_y = \bar{a}_z$ $\quad \bar{a}_y \times \bar{a}_z = \bar{a}_x$ $\quad \bar{a}_z \times \bar{a}_x = \bar{a}_y$

Consider a general vector $\bar{A}$ with components A_x, A_y, A_z along X, Y, Z respectively, then it can be represented in Cartesian coordinate system as

$$\bar{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$$

Here X ranges from $-\infty$ to ∞, Y from $-\infty$ to ∞, and Z from $-\infty$ to ∞.

***Note*:**

1. Differential displacement or elemental length is

$$d\bar{l} = dx\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z$$

2. Differential or elemental normal area is $d\bar{S} = dy\,dz\,\bar{a}_x$

$$= dx\,dz\,\bar{a}_y$$

$$= dx\,dy\,\bar{a}_z$$

3. Differential or elemental volume is $dv = dx\,dy\,dz$

1.4.2 Cylindrical Co-ordinate System

In this system ρ, ϕ and z are coordinates in which all are mutually orthogonal to each other.

Note: If the given problem is of circular symmetry, then it would be better to use cylindrical coordinates rather than Cartesian coordinates.

Where ρ is the radial distance from origin, ϕ is the azimuthal angle from X-axis to the radial distance and Z is same as in Cartesian coordinate system. The cylindrical coordinate system is shown in Fig. 1.6.

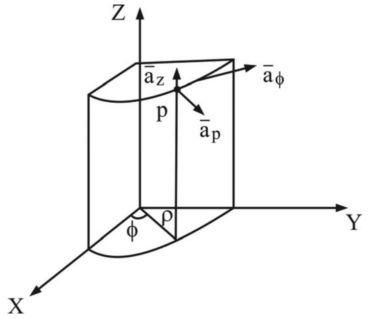


Fig. 1.6 Cylindrical coordinate system

Where $\bar{a}_\rho, \bar{a}_\phi$ and $\bar{a}_z$ are unit vectors along radial axis, azimuthal angle and z-direction respectively.

The dot product of any unit vector with itself gives '1'.

i.e. $\bar{a}_\rho \cdot \bar{a}_\rho = 1$ $\quad \bar{a}_\phi \cdot \bar{a}_\phi = 1$ $\quad \bar{a}_z \cdot \bar{a}_z = 1$

The dot product of any unit vector with the other unit vector gives '0'

i.e. $\bar{a}_\rho \cdot \bar{a}_\phi = 0$ $\quad \bar{a}_\phi . \bar{a}_z = 0$ $\quad \bar{a}_z \cdot \bar{a}_\rho = 0$

The cross product of any unit vector with the other unit vector, which is next to the first one in anticlockwise direction, results last unit vector in the anticlockwise direction.

i.e., $\bar{a}_\rho \times \bar{a}_\phi = \bar{a}_z$ $\quad \bar{a}_\phi \times \bar{a}_z = \bar{a}_\rho$ $\quad \bar{a}_z \times \bar{a}_\rho = \bar{a}_\phi$

Consider a general vector $\bar{A}$ with components A_ρ, A_ϕ, A_z along the three axes, then it can be represented as

$$\bar{A} = A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z$$

In this system $0 \le \rho < \infty$, $0 \le \phi < 2\pi$, and $-\infty < z < \infty$

The relation between Cylindrical and Cartesian coordinate system is shown in Fig.1.7.

The component of ρ on X-axis is ρ cosϕ and the component of ρ on Y-axis is ρ sin ϕ.

$\therefore \quad X = \rho \cos \phi, \quad Y = \rho \sin \phi, \quad Z = z$

from Fig.1.7 $\tan\phi = \frac{Y}{X} \Rightarrow \phi = \tan^{-1}\left(\frac{Y}{X}\right)$

$$X^2 + Y^2 = \rho^2 \Rightarrow \rho = \sqrt{X^2 + Y^2}$$

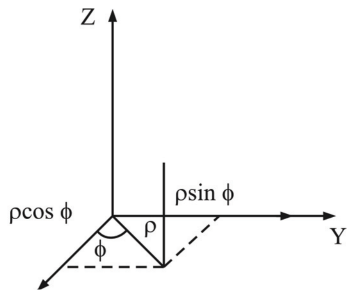


Fig.1.7 Relation between cylindrical and cartesian coordinate system

To find the relation among $\bar{a}_x$ and $\bar{a}_\rho, \bar{a}_\phi$ consider the Fig.1.8. A component of $\bar{a}_\rho$ on $\bar{a}_x$ is $\bar{a}_\rho \cos\phi$ and the component of $-\bar{a}_\phi$ on $\bar{a}_x$ is $-\bar{a}_\phi \sin\phi$.

$\therefore \quad \overline{a}_x$ can be written as $\overline{a}_x = \overline{a}_\rho \cos\phi - \overline{a}_\phi \sin\phi$

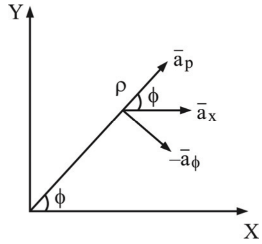


Fig. 1.8

To find the relation among $\overline{a}_y$ and $\overline{a}_\rho$, $\overline{a}_\phi$ consider the Fig.1.9. The component of $\overline{a}_\rho$ on $\overline{a}_y$ is $\overline{a}_\rho \sin\phi$ and the component of $\overline{a}_\phi$ on $\overline{a}_y$ is $\overline{a}_\phi \cos\phi$.

$\therefore \quad \overline{a}_y = \overline{a}_\rho \sin\phi + \overline{a}_\phi \cos\phi$

The unit vector $\overline{a}_z$ of Cartesian coordinate system and cylindrical coordinate system is same $\therefore \overline{a}_z = \overline{a}_z$

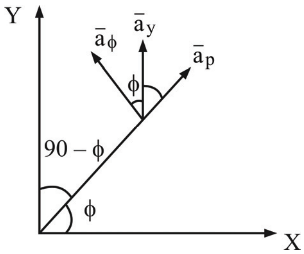


Fig. 1.9

We know that in Cartesian co-ordinate system

$$\overline{A} = A_x\overline{a}_x + A_y\overline{a}_y + A_z\overline{a}_z .$$

Substituting unit vectors,

$$\overline{A} = A_x\left(\overline{a}_\rho \cos\phi - \overline{a}_\phi \sin\phi\right) + A_y\left(\overline{a}_\rho \sin\phi + \overline{a}_\phi \cos\phi\right) + A_z\overline{a}_z$$

$$\overline{A} = \left(A_x \cos\phi + A_y \sin\phi\right)\overline{a}_\rho + \left(A_y \cos\phi - A_x \sin\phi\right)\overline{a}_\phi + A_z\overline{a}_z$$

$$\overline{A} = A_\rho\overline{a}_\rho + A_\phi\overline{a}_\phi + A_z\overline{a}_z$$

where

$A_\rho = A_x \cos\phi + A_y \sin\phi$

$A_\phi = -A_x \sin\phi + A_y \cos\phi$

$A_z = A_z$

i.e., in matrix form

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

The above matrix in terms of unit vectors is given by

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \overline{a}_\rho \cdot \overline{a}_x & \overline{a}_\rho \cdot \overline{a}_y & \overline{a}_\rho \cdot \overline{a}_z \\ \overline{a}_\phi \cdot \overline{a}_x & \overline{a}_\phi \cdot \overline{a}_y & \overline{a}_\phi \cdot \overline{a}_z \\ \overline{a}_z \cdot \overline{a}_x & \overline{a}_z \cdot \overline{a}_y & \overline{a}_z \cdot \overline{a}_z \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

or $$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \bar{a}_x \cdot \bar{a}_\rho & \bar{a}_x \cdot \bar{a}_\phi & \bar{a}_x \cdot \bar{a}_z \\ \bar{a}_y \cdot \bar{a}_\rho & \bar{a}_y \cdot \bar{a}_\phi & \bar{a}_y \cdot \bar{a}_z \\ \bar{a}_z \cdot \bar{a}_\rho & \bar{a}_z \cdot \bar{a}_\phi & \bar{a}_z \cdot \bar{a}_z \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Note:

1. Differential displacement or elemental length is
$$d\bar{l} = d\rho\,\bar{a}_\rho + \rho d\phi\,\bar{a}_\phi + dz\,\bar{a}_z$$
2. Differential or elemental normal area is $d\bar{S} = \rho d\phi\; dz\;\bar{a}_\rho$
$$= d\rho dz\,\bar{a}_\phi$$
$$= \rho d\phi\, d\rho\,\bar{a}_z$$
3. Differential or elemental volume is dv = $\rho d\rho\, d\phi\, dz$

1.4.3 Spherical Coordinate System

When the given problem is of spherical symmetry, it is better to use spherical coordinate system to solve the problem instead of either Cartesian or cylindrical coordinate system.

In this system r, θ, ϕ are coordinates in which all are mutually orthogonal to each other. Where 'r' is the distance from origin to the point (where the vector is located). θ is the co-latitude angle which is taken from z axis to the radial distance and ϕ is same as in cylindrical coordinate system.

The spherical coordinate system is shown in Fig.1.10. Where $\bar{a}_r$ is the unit vector along r, $\bar{a}_\theta$ is the unit vector in increasing direction of θ and $\bar{a}_\phi$ is the unit vector in increasing direction of ϕ.

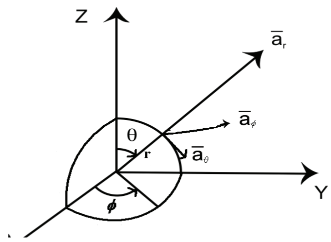


Fig. 1.10 Spherical coordinate system

The dot product of any unit vector with itself gives unity

i.e., $\bar{a}_r \cdot \bar{a}_r = 1 \quad \bar{a}_\theta \cdot \bar{a}_\theta = 1 \quad \bar{a}_\phi \cdot \bar{a}_\phi = 1$

The dot product of any unit vector with the other unit vector gives '0'

i.e., $\bar{a}_r \cdot \bar{a}_\theta = 0 \quad \bar{a}_\theta \cdot \bar{a}_\phi = 0 \quad \bar{a}_\phi \cdot \bar{a}_r = 0$

The cross product of unit vectors is: $\bar{a}_r \times \bar{a}_\theta = \bar{a}_\phi$, $\bar{a}_\theta \times \bar{a}_\phi = \bar{a}_r$, $\bar{a}_\phi \times \bar{a}_r = \bar{a}_\theta$

Here $0 \le r < \infty$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$.

To convert from Cartesian to cylindrical or spherical co-ordinate system consider Fig. 1.11.

The component of r on z-axis is $r \cos\, \theta$, and the component of r on ρ is $r \sin\theta$.

$\therefore \quad z = r\cos\theta$

$\rho = r\sin\theta$ and

we know $x = \rho\cos\phi$ & $y = \rho\sin\phi$

From Cartesian to cylindrical, the conversion is $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $z = z$

To get conversion from Cartesian to spherical co-ordinate system, substitute $\rho = r \sin\, \theta$ in the above equations.

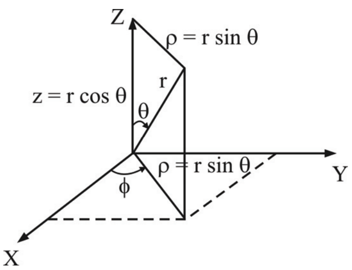


Fig. 1.11

$x = r\sin\theta\cos\phi$,

$y = r\sin\theta\sin\phi$, and

$z = r\cos\,\theta$

From the above equations $r = \sqrt{x^2 + y^2 + z^2}$

From Fig.1.11 $\quad \phi = \tan^{-1}\left(\dfrac{y}{x}\right)$

and $\quad \tan\theta = \left(\dfrac{\rho}{z}\right) = \dfrac{\sqrt{x^2 + y^2}}{z}$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$

relation between unit vectors of Cartesian and spherical co-ordinate systems is as follows:

$\bar{a}_x = \sin\,\theta\cos\,\phi\bar{a}_r + \cos\,\theta\cos\,\phi\bar{a}_\theta - \sin\,\phi\,\bar{a}_\phi$

$\bar{a}_y = \sin\,\theta\sin\phi\bar{a}_r + \cos\,\theta\sin\phi\bar{a}_\theta + \cos\,\phi\,\bar{a}_\phi$

$\bar{a}_z = \cos\,\theta\,\bar{a}_r - \sin\,\theta\,\bar{a}_\theta$

***Note*:**

1. Differential displacement or elemental length is

$$d\bar{l} = dr\bar{a}_r + rd\theta\bar{a}_\theta + r\sin\theta d\phi\bar{a}_\phi$$

2. Differential or elemental normal area is

$$d\overline{S} = r^2 \sin\theta\, d\theta\, d\phi\, \overline{a}_r$$

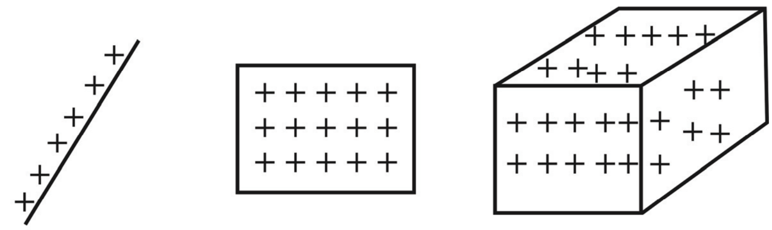
$$= r \sin\theta dr d\phi \overline{a}_\theta$$

$$= r dr d\theta \overline{a}_\phi$$

3. Differential or elemental volume is $dv = r^2 \sin\theta\, dr d\theta d\phi$

1.5 Electric Fields due to Continuous Charge Distributions

So far we have discussed the electric field or force due to point charges. Let us see the electric field due to continuous charge distribution along a line, on a surface and in a volume. If the charge is distributed along a line the distribution can be represented with the line charge density ρ_L(C/m), which is shown in Fig.1.12(a). If the charge is distributed on a surface it's distribution can be represented with the surface charge density ρ_s(C/m^2), which is shown in Fig. 1.12(b). If the charge is distributed in a volume it's distribution can be represented with the volume charge density ρ_v(C/m^3), which is shown in Fig. 1.12(c).



(a) a line charge (b) surface charge (c) volume charge

Fig.1.12 Charge distribution

The elemental charge dQ along a line can be written as

$dQ = \rho_l dl$, where dl is the elemental length.

So $$Q = \int_l \rho_l \, dl$$

∴ Electric field intensity due to line charge distribution is

$$\overline{E} = \int_l \frac{\rho_l \, dl}{4\pi \epsilon_0 R^2} \overline{a}_R \qquad \text{.....(1.5.1)}$$

The elemental charge dQ on a surface can be written as $dQ = \rho_s ds$

$\Rightarrow \qquad Q = \int_S \rho_s ds$

$\therefore$ Electric field intensity due to surface charge distribution is

$$\bar{E} = \int_s \frac{\rho_s\, ds}{4\pi \in_0 R^2} \bar{a}_R \qquad \text{.....(1.5.2)}$$

The elemental charge dQ in a volume can be written as $dQ = \rho_v dv$

$\Rightarrow \qquad Q = \int_v \rho_v dv$

$\therefore$ Electric field intensity due to volume charge distribution is

$$\bar{E} = \int_v \frac{\rho_v\, dv}{4\pi \in_0 R^2} \bar{a}_R \qquad \text{.....(1.5.3)}$$

1.5.1 Line Charge Distribution

Consider a line charge distribution from A to B along Z-axis as shown in Fig.1.13.

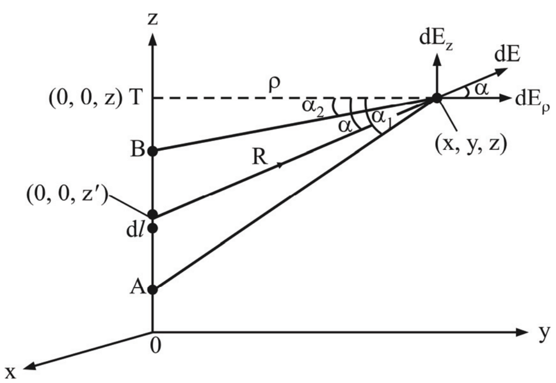


Fig.1.13 Finding $\bar{E}$ due to line charge distribution

Let us find the electric field at point (x, y, z) due to line charge distribution along Z- axis. We know electric field intensity due to line charge distribution as

$$\bar{E} = \int_l \frac{\rho_l\, dl}{4\pi \in_0 R^2} \bar{a}_R$$

where $\qquad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|}$, $dl = dz'$, Since the charge distribution has cylindrical symmetry, we use cylindrical coordinate system to obtain Electric field intensity.

From the Fig. 1.13

$$\bar{R} = \rho \bar{a}_\rho + (z - z')\bar{a}_z$$

$$\bar{E} = \int_l \frac{\rho_L \, dz'}{4\pi \in_0 R^2} \frac{\bar{R}}{|\bar{R}|}$$

$$= \int_l \frac{\rho_L \, dz'}{4\pi \in_0} \frac{\bar{R}}{R^3} = \int_l \frac{\rho_L \, dz'}{4\pi \in_0} \frac{\left[\rho \bar{a}_\rho + (z - z')\bar{a}_z\right]}{\left(\rho^2 + (z - z')^2\right)^{3/2}}$$

$$= \int_l \frac{\rho_L \, dz'}{4\pi \in_0} \frac{\left[\rho \bar{a}_\rho + (z - z')\bar{a}_z\right]}{\left(\rho^2 + (z - z')^2\right)^{3/2}}$$

From the Fig. 1.13

$$\tan\alpha = \frac{z - z'}{\rho} \Rightarrow z - z' = \rho \tan\alpha$$

$$\cos\alpha = \frac{\rho}{R} \Rightarrow R = \rho \sec\alpha \Rightarrow \sqrt{\rho^2 + \left(z - z'\right)^2} = \rho \sec\alpha$$

$$z' = \text{OT} - (z - z') = \text{OT} - \rho \tan\alpha$$

$$d\,z' = 0 - \rho \sec^2\alpha \, d\alpha$$

$$\bar{E} = \frac{-\rho_L}{4\pi \in_0} \int_l \frac{\rho \sec^2 \alpha \, d\alpha \left(\rho \bar{a}_\rho + \rho \tan\alpha \; \bar{a}_z\right)}{\rho^3 \sec^3 \alpha}$$

$$= \frac{-\rho_L}{4\pi \in_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho \sec^2 \alpha \, d\alpha \;\; \rho \sec\alpha \left(\bar{a}_\rho \cos\alpha + \bar{a}_z \sin\alpha\right)}{\rho^3 \sec^3 \alpha}$$

$$= \frac{-\rho_L}{4\pi \in_0 \rho} \int_{\alpha_1}^{\alpha_2} (\cos\alpha \; \bar{a}_\rho + \sin\alpha \, \bar{a}_z) \; d\alpha$$

$$= \frac{-\rho_L}{4\pi \in_0 \rho} \left[\left[\sin\alpha \; \bar{a}_\rho\right]_{\alpha_1}^{\alpha_2} - \left[\cos\alpha \, \bar{a}_z\right]_{\alpha_1}^{\alpha_2} \right]$$

$$= \frac{-\rho_L}{4\pi \in_0 \rho}\left[\left(\sin\alpha_2 - \sin\alpha_1\right)\bar{a}_\rho + \left(-\cos\alpha_2 + \cos\alpha_1\right)\bar{a}_z\right]$$

which is electric field at point (x, y, z) due to line charge distribution from ‘A’ to ‘B’ along Z-axis. If ‘A’ is tending to $-\infty$ then α_1 becomes $\pi/2$ and ‘B’ is tending to ∞ then α_2 becomes $-\pi/2$.

$$\bar{E} = \frac{-\rho_L}{4\pi \in_0 \rho}\left[\left(\sin\left(-\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right)\bar{a}_\rho + \left(-\cos\left(-\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right)\bar{a}_z\right]$$

$$= \frac{2\bar{a}_\rho \rho_L}{4\pi \in_0 \rho}$$

$$\bar{E} = \frac{\rho_L}{2\pi \in_0 \rho}\bar{a}_\rho \qquad \text{.....(1.5.4)}$$

which is the electric field at point (*x, y, z*) due to infinite line charge distribution along Z-axis.

1.5.2 Surface Charge Distribution

Consider an infinite sheet lying on XY plane which is perpendicular to Z-axis as shown in the Fig. 1.14.

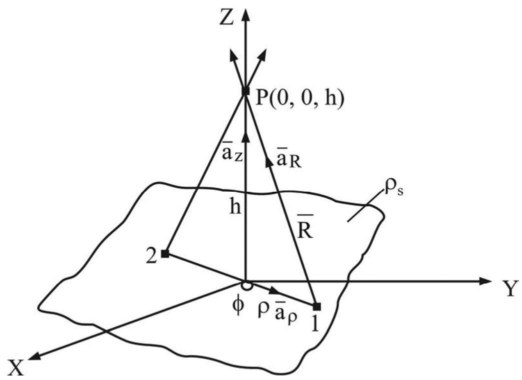


Fig. 1.14 Finding $\bar{E}$ due to infinite sheet of charge

Assume that the elemental surfaces are located on the sheet at ‘1’ and ‘2’ .

Then the elemental charge dQ on elemental surface ds is $dQ = \rho_s\, ds$.

$\therefore$ The elemental electric field at point (0, 0, h) due to the elemental surface ds is

$$d\overline{E} = \frac{dQ}{4\pi \in_0 R^2}\overline{a}_R$$

where

$$dQ = \rho_s ds \text{ and } \overline{a}_R = \frac{\overline{R}}{|\overline{R}|}$$

Since the surface is infinite it has circular symmetry, hence we can use cylindrical coordinate system to obtain electric field intensity.

Here ds lies on ρ and ϕ axises, Hence $ds = d\rho\, \rho\, d\phi$

From Fig.1.14

$$\rho\overline{a}_\rho + \overline{R} = h\overline{a}_z$$

$$\Rightarrow \quad \overline{R} = h\overline{a}_z - \rho\overline{a}_\rho$$

$$\therefore \quad d\overline{E} = \frac{dQ}{4\pi \in_0}\frac{\overline{R}}{|\overline{R}|^3}$$

$$= \frac{dQ}{4\pi \in_0}\frac{-\rho\overline{a}_\rho + h\overline{a}_z}{\left(\rho^2 + h^2\right)^{3/2}}$$

Since the sheet is symmetry with respect to origin on XY plane, for every electric field due to elemental surface (for example elemental surface located at '1') there will be an equal and opposite electric field due to the elemental surface on the other side(for example elemental surface located at '2') in the direction of 'ρ' (radial length), so finally when we add up the electric fields due to all the elemental surfaces on the sheet the electric field in the 'ρ' direction will get cancelled. We will have only the electric field perpendicular to the sheet i.e., along Z-direction.

By integrating the above equation, $\overline{E} = \dfrac{Q}{4\pi \in_0}\dfrac{h\overline{a}_z}{\left(\rho^2 + h^2\right)^{3/2}}$

Where $$Q = \int_\phi\int_\rho \rho_s\, \rho\, d\rho\, d\phi$$

$$\therefore \quad \overline{E} = \int_0^{2\pi}\int_0^{\infty}\frac{h\overline{a}_z}{\left(\rho^2 + h^2\right)^{3/2}}\rho_s\, \rho\, d\rho\, d\phi \frac{1}{4\pi \in_0}$$

$$\bar{E}=\frac{\rho_s}{4\pi\in_0}\int_0^{2\pi}d\phi\int_0^{\infty}\frac{h\rho}{\left(\rho^2+h^2\right)^{3/2}}d\rho\,\bar{a}_z$$

$$=\frac{\rho_s h}{4\pi\in_0}(2\pi)\int_0^{\infty}\left(\rho^2+h^2\right)^{-3/2}\frac{1}{2}d\left(\rho^2\right)\bar{a}_z$$

$$=\frac{\rho_s h}{2\in_0}\frac{1}{2}\left[\frac{\left(\rho^2+h^2\right)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\bar{a}_z\right]_0^{\infty}$$

$$\bar{E}=\frac{\rho_s h}{2\in_0}\frac{1}{2}\left[\frac{-\left(h^2\right)^{-1/2}}{-1/2}\right]\bar{a}_z$$

$$\bar{E}=\frac{\rho_s}{2\in_0}\bar{a}_z \qquad .....(1.5.5)$$

If we observe the above equation, the electric field is independent of the height ‘h’ i.e., the point can be considered at anywhere on the Z-axis.

The above equation can be generalized as

$$\bar{E}=\frac{\rho_s}{2\in_0}\bar{a}_n \qquad .....(1.5.6)$$

Where $\bar{a}_n$ is the unit vector which is perpendicular to the sheet.

Consider a parallel plate capacitor of equal and opposite charge on each plate, the electric field due to these parallel plates can be written as

$$\bar{E}=\frac{\rho_s}{2\in_0}\bar{a}_n+\frac{(-\rho_s)}{2\in_0}(-\bar{a}_n)=\frac{\rho_s}{\in_0}\bar{a}_n \qquad .....(1.5.7)$$

Problem 1.8

A circular ring of radius ‘a’ carries a uniform charge ρ_L C/m and is placed on the XY plane with axis the same as the Z-axis.

(a) Show that $\bar{E}(0,0,h)=\dfrac{\rho_L ah}{2\in_0\left(h^2+a^2\right)^{3/2}}\bar{a}_z$.

(b) What values of h gives the maximum value of $\bar{E}$

(c) If the total charge on the ring is Q. Find $\bar{E}$ as ‘a’ tends to zero.

Solution

(a) Here $$dl = a\, d\phi$$

$$dQ = \rho_L\, dl$$

$$= \rho_L\, a d\phi$$

$$\therefore \quad d\bar{E} = \frac{dQ}{4\pi \in_0 R^2}\bar{a}_r$$

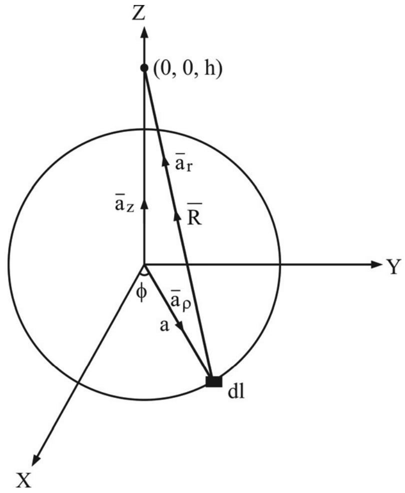


Fig. 1.15

$$\bar{a}_r = \frac{\bar{R}}{|R|}; \ \frac{\bar{a}_r}{R^2} = \frac{\bar{R}}{R^3}$$

$$\therefore \quad d\bar{E} = \frac{dQ}{4\pi \in_0} \frac{\left[-a\,\bar{a}_\rho + h\bar{a}_z\right]}{\left(a^2 + h^2\right)^{3/2}}$$

$$dQ = \rho_L\, a\, d\phi$$

$$Q = \int \rho_L\, a\, d\phi$$

when we add up electric fields, the electric field in ρ direction gets cancelled.

$$\therefore \quad \bar{E} = \frac{dQ}{4\pi \in_0} \frac{h\bar{a}_z}{\left(a^2 + h^2\right)^{3/2}}$$

$$= \int \frac{\rho_L a\, d\phi}{4\pi \in_0} \frac{h\bar{a}_z}{\left(a^2 + h^2\right)^{3/2}}$$

$$= \frac{\rho_L a}{4\pi \in_0} \frac{h\bar{a}_z}{\left(a^2 + h^2\right)^{3/2}} \int_0^{2\pi} d\phi = \frac{\rho_L ah}{2 \in_0 \left(a^2 + h^2\right)^{3/2}} \bar{a}_z$$

(b) $\frac{d\bar{E}}{dh} = 0$

$$\frac{\rho_L a}{2 \in_0} \bar{a}_z \frac{\left(a^2 + h^2\right)^{3/2} .1 - h \frac{3}{2} (a^2 + h^2)^{1/2} 2h}{\left(a^2 + h^2\right)^3} = 0$$

$(a^2 + h^2) - 3h^2 = 0$

$a^2 - 2h^2 = 0$

$2h^2 = a^2$

$$h = \pm \frac{a}{\sqrt{2}}$$

(c) When 'a' tends to zero, it becomes a point charge 'Q' located at origin and we have to find electric field at (0, 0, h) due to point charge 'Q' located at origin.

$$\therefore \qquad \bar{E} = \frac{Q}{4\pi \in_0 h^2} \bar{a}_z$$

Problem 1.9

Derive an expression for the electric field strength due to a circular ring of radius 'a' and uniform charge density ρ_L C/m. Obtain the value of height 'h' along Z-axis at which the net electric field becomes zero. Assume the ring to be placed in X-Y plane.

Solution

Derivation is as in Problem. 1.8.

$$\bar{E} = \frac{\rho_L ah}{2 \in_0 \left(a^2 + h^2\right)^{3/2}} \bar{a}_z$$

Which can be written as

$$\bar{E} = \frac{\rho_L a}{2 \in_0 h^2 \left(\frac{a^2}{h^2} + 1 \right)^{3/2}} \bar{a}_z$$

From the above equation we can say that for $h = \infty$, the net electric field becomes zero.

Problem 1.10

A circular ring of radius 'a' carries uniform charge ρ_L C/m and is in XY-plane. Find the Electric field at point (0, 0, 2) along its axis.

Solution

Replacing 'h' in problem.1.8 with '2' and solving, we get

$$\bar{E} = \frac{\rho_L a^2}{2 \in_0 \left(a^2 + 4 \right)^{3/2}} \bar{a}_z$$

1.5.3 Volume Charge Distribution

Consider a sphere of radius '*a*' as shown in the Fig.1.16.

Assume elemental volume *dv* is placed at point (r', θ', ϕ'). The elemental charge *dQ* due to the elemental volume *dv*, whose volume charge density ρ_v is

$$dQ = \rho_v dv$$

$$Q = \rho_v \int_v dv$$

$$= \rho_v \frac{4}{3} \pi a^3$$

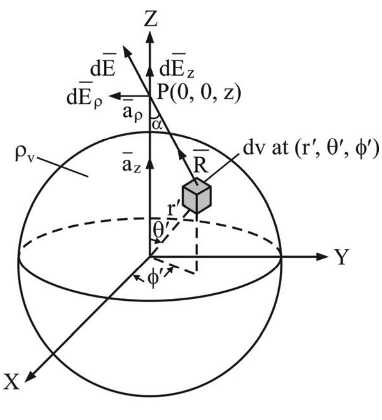


Fig. 1.16 Finding $\bar{E}$ due to volume charge distribution

The elemental electric field $d\overline{E}$ due to elemental volume dv is

$$d\overline{E} = \frac{dQ}{4\pi \in_0 R^2}\overline{a}_R$$

$$= \frac{\rho_v dv}{4\pi \in_0 R^2}\overline{a}_R$$

where $\overline{a}_R = \cos\alpha\,\overline{a}_z + \sin\alpha\,\overline{a}_\rho$

Due to symmetry, the electric field in 'ρ' direction will be zero. Finally total electric field will be in Z-direction.

$$\overline{E}_z = \overline{E}\cdot\overline{a}_z = \int_v \frac{\rho_v dv}{4\pi \in_0 R^2}\cos\alpha$$

In spherical coordinate system

$$dv = dr'\,r'\,d\theta'\,r'\sin\theta' d\phi'$$

$$dv = (r')^2 \sin\theta' dr' d\theta' d\phi'$$

$$\overline{E}_z = \int_v \frac{\rho_v (r')^2 \sin\theta' dr' d\theta' d\phi' \cos\alpha}{4\pi \in_0 R^2}$$

By applying cosine rule in the Fig.1.16

$$(r')^2 = z^2 + R^2 - 2zR\cos\alpha$$

$$\cos\alpha = \frac{-(r')^2 + z^2 + R^2}{2zR}$$

Similarly

$$R^2 = z^2 + (r')^2 - 2zr'\cos\theta'$$

$$\Rightarrow \qquad \cos\theta' = \frac{z^2 + (r')^2 - R^2}{2zr'} \qquad \text{.....(1.5.8)}$$

On differentiating equation (1.5.8), we get

$$-\sin\theta'\,d\theta' = \frac{-2R}{2zr'}dR$$

$$\sin\theta'\,d\theta' = \frac{R}{zr'}dR$$

Here as θ' varies from 0 to π, R changes from $z - r'$ to $z + r'$ respectively

Substituting $\cos\alpha$ and $\sin\theta' d\theta'$ in $\bar{E}_z$ equation, we get

$$\bar{E}_z = \frac{\rho_v}{4\pi \in_0} \int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r'^2 \frac{RdR}{zr'} dr' \frac{z^2 + R^2 - r'^2}{2zR} \frac{1}{R^2}$$

$$\bar{E}_z = \frac{\rho_v 2\pi}{8\pi \in_0 z^2} \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r' \left[1 + \frac{z^2 r'^2}{R^2}\right] dR\, dr'$$

$$\bar{E}_z = \frac{\rho_v \pi}{4\pi \in_0 z^2} \int_{r'=0}^{a} r' \left[R - \frac{z^2 - r'^2}{R}\right]_{z-r'}^{z+r'} dr'$$

$$\bar{E}_z = \frac{\rho_v \pi}{4\pi \in_0 z^2} \int_{r'=0}^{a} 4r'^2 dr'$$

$$\bar{E}_z = \frac{\rho_v}{\in_0 z^2} \frac{a^3}{3} = \frac{\rho_v}{4\pi \in_0 z^2} \frac{4}{3} \pi a^3$$

$$\bar{E} = \frac{Q}{4\pi \in_0 z^2} \bar{a}_z \qquad \text{.....(1.5.9)}$$

The electric field due to a sphere of radius 'a' with volume charge density ρ_v is similar to the electric field due to a point charge which is placed at origin.

Problem 1.11

A circular disk of radius 'a' is uniformly charged with ρ_s C/m^2. If the disk lies on the Z = 0 plane with it's axis along the Z-axis

(a) Show that at point (0, 0, h), $\bar{E} = \frac{\rho_s}{2 \in_0} \left[1 - \frac{h}{\left(h^2 + a^2\right)^{1/2}}\right] \bar{a}_z$

(b) From this derive the $\bar{E}$ due to an infinite sheet of charge on the Z = 0 plane.

(c) If $a << h$, Show that $\bar{E}$ is similar to the field due to a point charge.

Solution

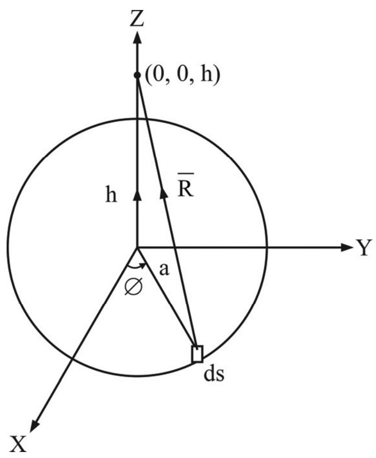


Fig. 1.17

(a) $$d\overline{E} = \frac{dQ}{4\pi \in_0 R^2}\overline{a}_r$$

$$dQ = \rho_s\, ds; \quad ds = d\rho.\rho\, d\phi,$$

$$= \rho_s\, \rho d\rho\, d\phi$$

$$\rho\overline{a}_\rho + \overline{R} = h\overline{a}_z$$

$$\overline{R} = h\overline{a}_z - \rho\overline{a}_\rho$$

$$\overline{E} = \int_s \frac{\rho_s \rho\, d\rho\, d\phi}{4\pi \in_0}\frac{\left(h\overline{a}_z - \rho\overline{a}_\rho\right)}{\left(h^2+\rho^2\right)^{3/2}}$$

$$\overline{E} = \frac{\rho_s}{4\pi \in_0}\overline{a}_z \int_0^{2\pi} d\phi \int_0^a \frac{\rho h}{\left(h^2+\rho^2\right)^{3/2}} d\rho$$

$$= \frac{\rho_s}{4\pi \in_0}\overline{a}_z\, 2\pi h \int_0^a \frac{1}{2}\left(h^2+\rho^2\right)^{-3/2} d\left(\rho^2\right)$$

$$= \frac{\rho_s h}{2 \in_0} \bar{a}_z \frac{1}{2} \left[\frac{\left(h^2 + \rho^2\right)^{\frac{-3}{2}+1}}{\frac{-3}{2}+1} \right]_0^a$$

$$= \frac{\rho_s h}{4 \in_0} \bar{a}_z \left\{ -2\left[\left(h^2 + a^2\right)^{-1/2} - \left(h^2\right)^{-1/2} \right] \right\}$$

$$= \frac{-\rho_s h \bar{a}_z}{2 \in_0} \left[\frac{1}{\sqrt{\left(h^2 + a^2\right)}} - \frac{1}{h} \right]$$

$$\bar{E} = \frac{\rho_s}{2 \in_0} \left[1 - \frac{h}{\left(h^2 + a^2\right)^{1/2}} \right] \bar{a}_z$$

(b) $a \to \infty$;

$$\therefore \qquad \bar{E} = \frac{\rho_s}{2 \in_0} \bar{a}_z$$

(c) when a<< h, the volume charge density becomes a point charge located at origin,

$$\therefore \qquad \bar{E} = \frac{Q}{4\pi \in_0 h^2} \bar{a}_z$$

Problem 1.12

The finite sheet $0 < x < 1$, $0 < y < 1$ on the Z = 0 plane has a charge density $\rho_s = xy\,(x^2 + y^2 + 25)^{3/2}$ nC/m^2.

Find

(a) the total charge on the sheet

(b) the electric field at (0, 0, 5)

(c) the force experienced by a – 1 nC charge located at (0, 0, 5)

Solution

(a) $dQ = \rho_s\, ds$

$$Q = \int_s \rho_s ds$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} xy\left(x^2+y^2+25\right)^{3/2} n\,dx\,dy$$

$$= n\int_{x=0}^{1} x\int_{y=0}^{1}\left(x^2+y^2+25\right)^{3/2}\frac{1}{2}d(y^2)dx$$

$$= n\int_{x=0}^{1} x\left[\left(x^2+y^2+25\right)^{5/2}\right]_0^1\frac{2}{5}\frac{1}{2}dx$$

$$= \frac{n}{5}\int_{x=0}^{1}\left[\left(x^2+26\right)^{5/2}-\left(x^2+25\right)^{5/2}\right]\frac{1}{2}d(x^2)$$

$$= \frac{n}{5}\left[\left(x^2+26\right)^{7/2}-\left(x^2+25\right)^{7/2}\right]_0^1\frac{1}{7}$$

$$= \frac{n}{35}\left[(27)^{7/2}-2(26)^{7/2}+(25)^{7/2}\right]$$

$$= \frac{n}{35}\left[102275.868136-179240.733942+78125\right]$$

Q = 33.15 nC

(b) Electric field at (0, 0, 5)

$$d\bar{E} = \frac{\rho_s ds}{4\pi\in_0 R^2}\bar{a}_R\,; \qquad \text{on Z-plane point is } (x, y, 0)$$

$$\therefore \quad \bar{R} = (0,0,5)-(x,y,0) = -x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z$$

$$\frac{\bar{a}_R}{R^2} = \frac{\bar{R}}{|\bar{R}|^3} = \frac{-x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z}{\left(\sqrt{x^2+y^2+25}\right)^3}$$

$$\bar{E} = \int_s \frac{\rho_s ds}{4\pi\in_0}\frac{\bar{R}}{|\bar{R}|^3}$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\frac{xy\left(x^2+y^2+25\right)^{3/2}\times 10^{-9}}{4\pi\in_0}\left(\frac{-x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z}{\left(\sqrt{x^2+y^2+25}\right)^3}\right)dx\,dy$$

$$= \frac{1}{4\pi \in_0} \int_{x=0}^{1} \int_{y=0}^{1} -x^2 y\overline{a}_x - xy^2\overline{a}_y + 5xy\overline{a}_z dx\, dy \times 10^{-9}$$

$$= \frac{1}{4\pi \in_0} \int_{x=0}^{1} -x^2 \left[\frac{y^2}{2}\right]_0^1 \overline{a}_x - x\left[\frac{y^3}{3}\right]_0^1 \overline{a}_y + 5x\left[\frac{y^2}{2}\right]_0^1 \overline{a}_z\, dx \times 10^{-9}$$

$$= \frac{1}{4\pi \in_0} \int_{x=0}^{1} -\frac{x^2}{2}\overline{a}_x - \frac{x}{3}\overline{a}_y + \frac{5}{2}x\overline{a}_z\, dx \times 10^{-9}$$

$$= \frac{1}{4\pi \in_0}\left[\left[-\frac{x^3}{6}\right]_0^1 \overline{a}_x - \left[\frac{x^2}{6}\right]_0^1 \overline{a}_y + \frac{5}{2}\left[\frac{x^2}{2}\right]_0^1 \overline{a}_z\right] \times 10^{-9}$$

$$= \frac{1}{4\pi \in_0}\left[-\frac{1}{6}\overline{a}_x - \frac{1}{6}\overline{a}_y + \frac{5}{4}\overline{a}_z\right] \times 10^{-9}$$

$$= 9 \times 10^9 \left[-\frac{1}{6}\overline{a}_x - \frac{1}{6}\overline{a}_y + \frac{5}{4}\overline{a}_z\right] \times 10^{-9}$$

$$= -1.5\overline{a}_x - 1.5\overline{a}_y + 11.25\overline{a}_z \text{ V/m}$$

(c) $\overline{F} = q\overline{E}$

$$= (-1\, nC)\left[-1.5\overline{a}_x - 1.5\overline{a}_y + 11.25\overline{a}_z\right]$$

$$= 1.5\overline{a}_x + 1.5\overline{a}_y - 11.25\overline{a}_z \text{ nN}$$

Problem 1.13

A square plane described by $-2 < x < 2$, $-2 < y < 2$, $z = 0$ carries a charge density $12|y|$ mC/m^2. Find the total charge on the plate and the electric field intensity at (0, 0, 10)

Solution

$$dQ = \rho_s\, ds$$

$$Q = \int_s \rho_s ds$$

$$= \int_{x=-2}^{2} \int_{y=-2}^{2} 12\,|\,y\,| \times 10^{-3} dxdy$$

$$= 10^{-3} \int_{x=-2}^{2} \left[\int_{y=-2}^{0} -12y\,dy + \int_{y=0}^{2} 12y\,dy \right] dx$$

$$= 10^{-3} \int_{x=-2}^{2} -12\left[\frac{y^2}{2}\right]_{-2}^{0} + 12\left[\frac{y^2}{2}\right]_{0}^{2} dx$$

$$= 10^{-3} \int_{x=-2}^{2} 12(2)+12(2)\,dx$$

$$= 48\times 10^{-3} \int_{x=-2}^{2} dx = 48\times 10^{-3} \times 4 = 192 \text{ mC}$$

$$d\overline{E} = \frac{\rho_s ds}{4\pi \in_0 R^2}\overline{a}_R \, ; \quad \overline{R} = (0, 0, 10) - (x, y, 0) = -x\overline{a}_x - y\overline{a}_y + 10\overline{a}_z$$

$$d\overline{E} = \frac{\rho_s ds}{4\pi \in_0} \frac{\overline{R}}{R^3}$$

$$\overline{E} = \int_s \frac{\rho_s ds}{4\pi \in_0} \frac{\overline{R}}{R^3}$$

$$= \int_{x=-2}^{2} \int_{y=-2}^{2} \frac{12\,|\,y\,|\times 10^{-3}}{4\pi \in_0} \left(\frac{-x\overline{a}_x - y\overline{a}_y + 10\overline{a}_z}{\left(\sqrt{x^2 + y^2 + 100}\right)^3} \right) dx\,dy$$

$$= 9\times 10^6 \times 12 \int_{x=-2}^{2} \left[\int_{y=-2}^{0} \frac{xy\overline{a}_x + y^2\overline{a}_y - 10y\overline{a}_z}{\left(x^2 + y^2 + 100\right)^{3/2}} dy + \int_{y=0}^{2} \frac{-xy\overline{a}_x - y^2\overline{a}_y + 10y\overline{a}_z}{\left(x^2 + y^2 + 100\right)^{3/2}} dy \right] dx$$

Replacing y with –y in the first integral and simplifying

$$\overline{E} = 108\times 10^6 \int_{x=-2}^{2} \left[\int_{y=0}^{2} \frac{-2xy\overline{a}_x + 20y\overline{a}_z}{\left(x^2 + y^2 + 100\right)^{3/2}} dy \right] dx$$

$$= 108 \times 10^6 \int_{x=-2}^{2} \left[-x \int_{y=0}^{2} 2y\bar{a}_x \left(x^2 + y^2 + 100\right)^{-3/2} dy + 10 \int_{y=0}^{2} 2y\bar{a}_z \left(x^2 + y^2 + 100\right)^{-3/2} dy \right] dx$$

$$= 108 \times 10^6 \int_{x=-2}^{2} \left[-x \int_{y=0}^{2} \bar{a}_x \left(x^2 + y^2 + 100\right)^{-3/2} d(y^2) + 10 \int_{y=0}^{2} \bar{a}_z \left(x^2 + y^2 + 100\right)^{-3/2} d(y^2) \right] dx$$

$$= 108 \times 10^6 \int_{x=-2}^{2} \left[-x \left[\frac{\left(x^2 + y^2 + 100\right)^{-1/2}}{-1/2} \right]_0^2 \bar{a}_x + 10 \left[\frac{\left(x^2 + y^2 + 100\right)^{-1/2}}{-1/2} \right]_0^2 \bar{a}_z \right] dx$$

$$= 108 \times 10^6 \int_{x=-2}^{2} \left\{ \left[2x\left(x^2 + 104\right)^{-1/2} - 2x\left(x^2 + 100\right)^{-1/2} \right] \bar{a}_x - 20\left[\left(x^2 + 104\right)^{-1/2} - \left(x^2 + 100\right)^{-1/2} \right] \bar{a}_z \right\} dx$$

$\because$ $x\left(x^2 + 104\right)^{-1/2}$ & $x\left(x^2 + 100\right)^{-1/2}$ are odd functions

and $\left(x^2 + 104\right)^{-1/2}$ & $\left(x^2 + 100\right)^{-1/2}$ are even functions

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{if } f \text{ is odd} \\ 2\int_{0}^{a} f(x)dx & \text{if } f \text{ is even} \end{cases}$$

$$\therefore \quad \bar{E} = -20 \times 108 \times 10^6 \times 2 \int_{x=0}^{2} \left[\frac{1}{\sqrt{x^2 + \left(\sqrt{104}\right)^2}} - \frac{1}{\sqrt{x^2 + 10^2}} \right] \bar{a}_z dx$$

$$= -40 \times 108 \times 10^6 \left[\sinh^{-1}\left(\frac{x}{\sqrt{104}} \right) - \sinh^{-1}\left(\frac{x}{10} \right) \right]_0^2 \bar{a}_z$$

$$= -40 \times 108 \times 10^6 \left[\sinh^{-1}\left(\frac{2}{\sqrt{104}} \right) - \sinh^{-1}\left(\frac{1}{5} \right) \right] \bar{a}_z$$

$$= -40 \times 108 \times 10^6 \left[0.19488 - 0.19869\right] \bar{a}_z$$

$\bar{E} = 16.46\ \bar{a}_z$ MV/m.

1.6 Electric Flux Density or Displacement Density

It is also called Electric displacement and to understand the concept of Electric flux density, one needs to know about line integral, surface integral and electric flux, which are explained as follows.

1.6.1 Line Integral

If a vector $\overline{A}$ is passing through a line as shown in the Fig.1.18. The line integral can be defined as the tangential component of vector $\overline{A}$ along the line, which can be written as

$$\int_L |\overline{A}| \cos\theta \, dL = \int_L \overline{A}.\overline{dL}$$

If a line is closed curve then the above integral can be written as $\oint_L \overline{A}.\overline{dl}$ which is called as contour line integral.

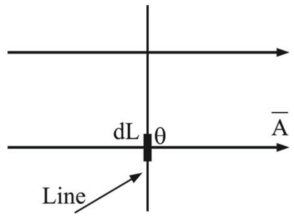


Fig. 1.18 Evaluation of line integral

1.6.2 Surface Integral

Similarly, if a vector $\overline{A}$ is passing through a surface as shown in Fig. 1.19

The flux (ψ) of a vector $\overline{A}$ or surface integral can be written as

$$\psi = \int_s |\overline{A}| \cos\theta \, ds$$

$$= \int_s \overline{A}.\overline{ds}$$

If it is closed surface then the above integral can be be written as $\oint_s \overline{A}.\overline{ds}$ which is called as contour surface integral.

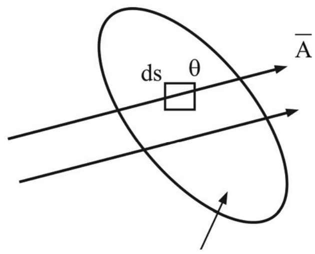


Fig. 1.19 Evaluation of surface integral

1.6.3 Electric Flux

We know that electric field intensity depends upon the medium in which it passes. Let us define a new vector $\overline{D}$ such that it is independent of medium i.e.,

$\overline{D} = \epsilon_0 \overline{E}$. Then the flux of $\overline{D}$, i.e., $\psi = \oint_S \overline{D}.\overline{ds}$, where ψ is the electric flux. Which can be defined according to SI units as one line of flux originates from +1 Coloumb and terminates at –1 Coloumb. So the unit of Electric flux is also Coloumb and $\overline{D}$ is the electric flux density whose unit is columb/m^2.

The formulae for $\bar{D}$ can be obtained by multiplying the formulae of $\bar{E}$ with $\in_0$.

$\therefore$ Electric flux density due to a point charge $\bar{D}_Q = \dfrac{Q}{4\pi}\dfrac{\bar{a}_R}{R^2}$ .....(1.6.1)

and Electric flux density due to an infinite line with line chare density

$$\rho_L \text{ is } \bar{D}_L = \frac{\rho_L}{2\pi\rho}\bar{a}_\rho \qquad (1.6.2)$$

Problem 1.14

Determine $\bar{D}$ at (4, 0, 3) if there is a point charge –5π mC at (4, 0, 0) and a line charge 3π mC/m along the Y-axis

Solution

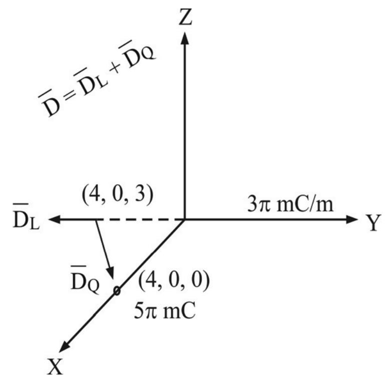


Fig. 1.20

$$\bar{D}_Q = \frac{Q}{4\pi}\frac{\bar{a}_R}{R^2}$$

where, $\bar{R} = (4,0,3) - (4,0,0) = (0,0,3)$

$$= \frac{-5\pi}{4\pi}\frac{3\bar{a}_z \times 10^{-3}}{(9)^{3/2}}$$

$$= \frac{-5}{4}\frac{3\bar{a}_z \times 10^{-3}}{27} = \frac{-5\bar{a}_z \times 10^{-3}}{36} = -0.139\bar{a}_z \times 10^{-3} \text{ C/m}^2$$

$$\bar{a}_\rho = \frac{\bar{\rho}}{|\bar{\rho}|}$$

$$\bar{\rho} = (4,0,3) - (0,0,0) = 4\bar{a}_x + 3\bar{a}_z$$

$$\bar{D}_L = \frac{\rho_L}{2\pi\rho}\bar{a}_\rho$$

$$= \frac{3\pi}{2\pi} \times 10^{-3}\frac{4\bar{a}_x + 3\bar{a}_z}{25}$$

$$= 0.24\bar{a}_x + 0.18\bar{a}_z \text{ mC/m}^2$$

$$\bar{D} = \bar{D}_Q + \bar{D}_L = 240\bar{a}_x + 42\bar{a}_z \ \mu\text{C/m}^2$$

1.7 Divergence of a Vector

Divergence: The divergence of a vector $\overline{A}$ at a given point is the outward flux in a volume as volume shrinks about the point. It can be represented as

$$div\,\overline{A} = \nabla \cdot \overline{A} = \lim_{\Delta v \to 0} \frac{\oint_S \overline{A} \cdot \overline{ds}}{\Delta v} \quad .....(1.7.1)$$

Where ∇ is the del operator or gradient operator. ∇ can be operated on a vector or scalar. It has got different meanings when it is operating on a vector and scalar. If it is operating on a scalar V then it can be written as ∇V which is called as scalar gradient. If it is operating on a vector $\overline{A}$ with dot product then it is $\nabla.\overline{A}$ and it is called as divergence of vector $\overline{A}$ and If it is operating on a vector $\overline{A}$ with cross product then it is $\nabla \times \overline{A}$ and it is called as curl of vector $\overline{A}$.

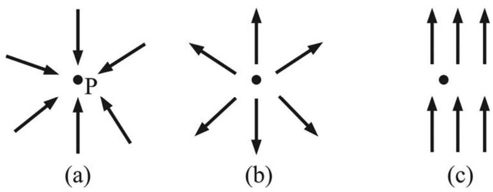


Fig. 1.21 Flux lines

Physically divergence can be interpreted as the measure of how much field diverges or emanates from a point. Let us consider the Fig.1.21(a) in which field is reaching to the point. Divergence at that point is –Ve or it is also called as convergence. In Fig.1.21(b) the field is going away from the point, therefore divergence is +Ve. In Fig.1.21(c) some of the flux lines or field lines are reaching to the point and same number of field lines are leaving from the point hence the divergence is zero.

To determine $\nabla.\overline{A}$ let us consider the volume in Cartesian co-ordinate systems as shown in the Fig.1.22. In Cartesian co-ordinate system, the vector $\overline{A}$ with it's unit vectors and components along X, Y, Z is

$$\overline{A} = A_x \overline{a}_x + A_y \overline{a}_y + A_z \overline{a}_z$$

Assume the elemental volume $\Delta V = \Delta x \Delta y \Delta z$. The flux of a vector $\overline{A}$ on Y-axis that enters in to the left side of the volume is $A_y \Delta x \Delta z$. The flux which is leaving from right side of the volume on Y-axis can be written as $(A_y + \Delta A_y)\,\Delta x \Delta z$. This equation can be modified as

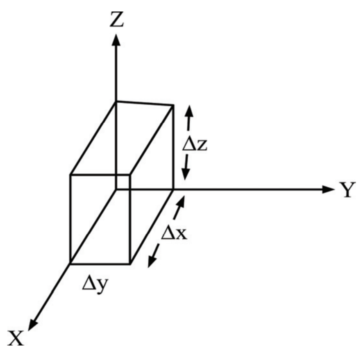


Fig. 1.22 Evaluation of $\nabla.\overline{A}$

$\left(A_y + \frac{\Delta A_y}{\Delta_y}\Delta_y\right)\Delta x\,\Delta z$. So the total flux on Y-axis is $A_y\Delta x\Delta z + \frac{\partial A_y}{\partial y}\Delta x\Delta y\Delta z - A_y\Delta x\Delta z$

$$= \frac{\partial A_y}{\partial y}\Delta x\Delta y\Delta z$$

Similarly on X and Z-axises also.

The entire flux in all the directions is $\psi = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta x\Delta y\Delta z$. We know $\psi = \oint_s \overline{A}\cdot d\overline{s}$

$$\frac{\oint_s \overline{A}\cdot d\overline{s}}{\Delta v} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Applying Limit on both sides

$$\lim_{\Delta v\to 0}\frac{\oint_s \overline{A}\cdot d\overline{s}}{\Delta v} = \lim_{\Delta v\to 0}\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\cdot\overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Conclusion

The divergence of a vector results a scalar. The divergence of a scalar has no meaning

$$\nabla\cdot\left(\overline{A} + \overline{B}\right) = \nabla.\overline{A} + \nabla.\overline{B}$$

$$\nabla\cdot\left(V\overline{A}\right) = V\nabla.\overline{A} + \overline{A}\cdot\nabla V$$

$$\nabla\cdot\overline{A} = \left(\frac{\partial\overline{a}_x}{\partial x} + \frac{\partial\overline{a}_y}{\partial y} + \frac{\partial\overline{a}_z}{\partial z}\right).\left(A_x\overline{a}_x + A_y\overline{a}_y + A_z\overline{a}_z\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

So from the above equation, the gradient operator is

$$\nabla = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \qquad \text{…..(1.7.2)}$$

and the scalar gradient is

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

1.7.1 Divergence Theorem

Statement

This theorem states that the outward flux flows through a closed surface is same as the volume integral of divergence of a vector.

$$\oint_s \bar{A} \cdot d\bar{s} = \int_v \nabla \cdot \bar{A}\, dv \qquad \text{….. (1.7.3)}$$

Proof:

Consider a vector $\bar{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$.

Similarly $d\bar{s} = ds_x\bar{a}_x + ds_y\bar{a}_y + ds_z\bar{a}_z$ and we know that divergence of vector $\bar{A}$ i.e.,

$$\nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Assume dv = dx dy dz

consider the volume integral

$$\int_v \nabla \cdot \bar{A}\, dv = \iiint_v \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dx\, dy\, dz$$

The second term in the above integral can be written as

$$\iiint_v \frac{\partial A_y}{\partial y} dx\, dy\, dz = \oiint_s \left[\int \frac{dA_y}{dy} dy \right] dx\, dz = \oiint_s A_y ds_y$$

where ds_y = The elemental surface on XZ plane.

Similarly the first and third terms can be written as

$$\oiint_s A_x ds_x \quad \text{and} \quad \oiint_s A_z ds_z$$

$$\therefore \quad \int_v \nabla \cdot \overline{A} dv = \oiint_s (A_x ds_x + A_y ds_y + A_z ds_z)$$

$$= \oiint_s \left(A_x \overline{a}_x + A_y \overline{a}_y + A_z \overline{a}_z\right) \cdot \left(ds_x \overline{a}_x + ds_y \overline{a}_y + ds_z \overline{a}_z\right) = \oiint_s \overline{A} \cdot \overline{ds}$$

Hence proved

Formulae for Gradient

in Cartesian co-ordinate system

$$\nabla V = \frac{\partial V}{\partial x}\overline{a}_x + \frac{\partial V}{\partial y}\overline{a}_y + \frac{\partial V}{\partial z}\overline{a}_z \qquad \text{.....(1.7.4)}$$

in cylindrical co-ordinate system

$$\nabla V = \frac{\partial V}{\partial \rho}\overline{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\overline{a}_\phi + \frac{\partial V}{\partial z}\overline{a}_z \qquad \text{.....(1.7.5)}$$

in spherical co-ordinate system

$$\nabla V = \frac{\partial V}{\partial r}\overline{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\overline{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\overline{a}_\phi \qquad \text{.....(1.7.6)}$$

Problem 1.15

Find the gradient of the following scalar fields

(a) $V = e^{-z} \sin 2x \cos hy$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10r \sin^2\theta \cos\phi$

Solution

(a) Since given V is in x and y, consider gradient in Cartesian co-ordinate system

$$\nabla V = \frac{\partial V}{\partial x}\overline{a}_x + \frac{\partial V}{\partial y}\overline{a}_y + \frac{\partial V}{\partial z}\overline{a}_z$$

$$= e^{-z}\cos h\, y \cos 2x\, 2\overline{a}_x + e^{-z}\sin 2x \sin h\, y\, \overline{a}_y + \sin 2x \cos h\, y\; e^{-z}\left(-1\right)\overline{a}_z$$

$$= 2\cos 2x \cos h\, y\, e^{-z}\overline{a}_x + \sin 2x \sin h\, y\, e^{-z}\,\overline{a}_y - \sin 2x \cos h\, y\; e^{-z}\overline{a}_z$$

(b) Since given U is in ρ, z and ϕ, consider gradient in cylindrical co-ordinate system

$$\nabla U = \frac{\partial U}{\partial \rho}\overline{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\overline{a}_\phi + \frac{\partial U}{\partial z}\overline{a}_z$$

$$= Z\cos 2\phi\, 2\rho\,\overline{a}_\rho + \rho z(-\sin 2\phi)2\,\overline{a}_\phi + \rho^2\cos 2\phi\,\overline{a}_z$$

(c) Since given W is in r, θ and ϕ, consider gradient in spherical co-ordinate system

$$\nabla W = \frac{\partial W}{\partial r}\overline{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\overline{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\overline{a}_\phi$$

$$= 10\sin^2\theta\cos\phi\,\overline{a}_r + \left(\frac{10r}{r}\right)2\sin\theta\cos\theta\cos\phi\,\overline{a}_\theta + 10r\sin^2\theta(-\sin\phi)\overline{a}_\phi \cdot \frac{1}{r\sin\theta}$$

Formulae for Divergence of a Vector

in Cartesian co-ordinate system

$$\nabla\cdot\overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \text{.....(1.7.7)}$$

in cylindrical co-ordinate system

$$\nabla\cdot\overline{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(A_\phi)}{\partial \phi} + \frac{\partial A_z}{\partial z} \qquad \text{.....(1.7.8)}$$

in spherical co-ordinate system

$$\nabla\cdot\overline{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \qquad \text{.....(1.7.9)}$$

Problem 1.16

Determine the divergence of the following vector fields.

(a) $\overline{P} = x^2yz\,\overline{a}_x + x^3zy\,\overline{a}_y + xy^2z^3\overline{a}_z$

(b) $\overline{Q} = \rho\sin\phi\,\overline{a}_\rho + \rho^2 z\,\overline{a}_\phi + z\cos\phi\,\overline{a}_z$

(c) $\overline{T} = \frac{1}{r^2}\cos\theta\,\overline{a}_r + r\sin\theta\cos\phi\,\overline{a}_\theta + \cos\theta\,\overline{a}_\phi$

(d) $\overline{N} = r^3\sin\theta\,\overline{a}_r + \sin 2\theta\cos^2\phi\,\overline{a}_\theta + \cos\theta\, r^2\,\overline{a}_\phi$

Solution

(a) Given $\overline{P} = x^2yz\,\overline{a}_x + x^3zy\,\overline{a}_y + xy^2z^3\overline{a}_z$

$$\nabla \cdot \overline{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

$$= 2xyz + x^3z + 3xy^2z^2$$

(b) Given $\overline{Q} = \rho \sin\phi \overline{a}_\rho + \rho^2 z \overline{a}_\phi + z\cos\phi \overline{a}_z$

$$\nabla \cdot \overline{Q} = \frac{1}{\rho}\frac{\partial(\rho Q_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(Q_\phi)}{\partial \phi} + \frac{\partial Q_z}{\partial z}$$

$$= \frac{1}{\rho} 2\rho \sin\phi + \frac{1}{\rho}(0) + \cos\phi$$

$$= 2\sin\phi + \cos\phi$$

(c) Given $\overline{T} = \frac{1}{r^2}\cos\theta \, \overline{a}_r + r\sin\theta\cos\phi \overline{a}_\theta + \cos\theta \overline{a}_\phi$

$$\nabla \cdot \overline{T} = \frac{1}{r^2}\frac{\partial(r^2 T_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta T_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_\phi}{\partial \phi}$$

$$= \frac{1}{r^2}(0) + \frac{1}{r\sin\theta} r \, 2\sin\theta\cos\theta\cos\phi + \frac{1}{r\sin\theta}(0)$$

$$= 2\cos\theta\cos\phi$$

(d) Given $\overline{N} = r^3 \sin\theta \, \overline{a}_r + \sin 2\theta \cos^2\phi \overline{a}_\theta + \cos\theta \, r^2 \, \overline{a}_\phi$

$$\nabla \cdot \overline{N} = \frac{1}{r^2}\frac{\partial(r^2 N_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta N_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial N_\phi}{\partial \phi}$$

$$= \frac{1}{r^2} 5r^4 \sin\theta + \frac{1}{r\sin\theta}\frac{1}{2}\left(-\sin\theta + \frac{\sin 3\theta}{3}\right)\cos^2\phi + \frac{1}{r\sin\theta}(0)$$

$$= 5r^2 \sin\theta - \frac{1}{2r}\cos^2\phi + \frac{\sin 3\theta}{6r\sin\theta}\cos^2\phi$$

1.8 Gauss's Law and Applications

1.8.1 Gauss Law

Gauss law states that the flux flowing through a closed surface is equivalent to the charge enclosed by that surface.

According to the statement $\psi = Q_{enc}$ .....(1.8.1)

Where ψ is the flux flowing through a closed surface. Q_{enc} is the charge enclosed by the closed surface.

We know $\psi = \oint_S \bar{D} \cdot d\bar{s}$

The charge enclosed within a volume or closed surface whose volume charge density ρ_v is

$$Q = \int_v \rho_v \, dv$$

According to Gauss's law we can write as

$$\psi = \oint_S \bar{D} \cdot d\bar{s} = \int_v \rho_v \, dv \quad .....(1.8.1a)$$

According to divergence theorem we can write

$$\oint_S \bar{D} \cdot d\bar{s} = \int_V \nabla \cdot \bar{D} \, dv \quad .....(1.8.1b)$$

By comparing the volume integrals in equations (1.8.1a) and (1.8.1b) we can write as

$$\rho_V = \nabla \cdot \bar{D} \quad .....(1.8.2)$$

which is the Maxwell's first equation for electrostatics (time in-varying fields)

Consider unsymmetrical distribution as shown in Fig. 1.23a. The flux flowing through the closed surface shown in Fig. 1.23a is $\psi = 5 - 2 = 3$ nC. The charge enclosed by the surface is $Q = 3$ nC.

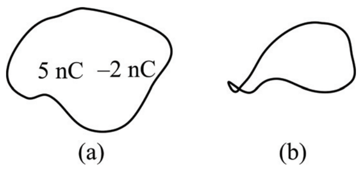


Fig. 1.23 Closed surface

Consider an empty closed surface as shown in Fig. 1.23b. Flux flowing through the closed surface shown in Fig. 1.23b is ψ = 0 and hence charge enclosed by the closed surface is zero.

Conclusion

Gauss law holds good even if the charge distribution is unsymmetrical as shown in Figs.1.23a & b. But to find either $\bar{E}$ or $\bar{D}$, the charge distribution must be symmetrical. It can be rectangular symmetry or cylindrical symmetry or spherical symmetry.

If the continuous charge distribution depends on either '*x*' or '*y*' or '*z*', then the distribution will have rectangular symmetry. So to find either $\bar{E}$ or $\bar{D}$, we can use rectangular co-ordinates.

If the continuous charge distribution depends only on ρ and is independent of ϕ and z then the distribution will have cylindrical symmetry. So, to find either $\bar{E}$ or $\bar{D}$, we can use cylindrical co-ordinates.

If the continuous charge distribution depends on 'r' and is independent of θ and ϕ then the symmetry it will have is spherical. So to find either $\bar{E}$ or $\bar{D}$, we can use spherical co-ordinates.

1.8.2 Applications of Gauss's Law – Point Charge

We need to find $\bar{D}$ at any point surrounded by Q. Assume that the point charge is located at origin, then a sphere can be assumed, that surrounds the point charge as shown in Fig.1.24, which shows the problem has spherical symmetry and spherical coordinate system can be used to obtain $\bar{D}$. Let us find out $\bar{D}$ at point 'P' due to a point charge.

The electric flux density $\bar{D}$ is normal or perpendicular to the spherical surface. i.e., $\bar{D} = D_r \bar{a}_r$.

The elemental surface *ds* lies on θ and ϕ axises. i.e., *ds* is normal to r axis.

$$\therefore \quad d\bar{s} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

Flux flowing through the sphere is

$$\psi = \oint_S \bar{D}.\overline{ds}$$

$$\therefore \quad \psi = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \bar{D}_r \bar{a}_r . r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$\psi = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \int_{\phi=0}^{2\pi} r^2 \left[-\cos\theta\right]_0^{\pi} d\phi$$

$$\psi = 2D_r r^2 \left[2\pi\right]$$

$$\psi = 4\pi r^2 D_r$$

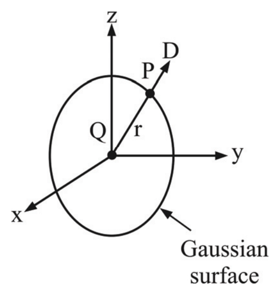


Fig. 1.24 Gaussian surface about a point charge

The charge enclosed by the sphere is

$$Q_{enc} = Q$$

According to Gauss's law

$$\psi = Q_{enc}$$

$\therefore$ $$Q = 4\pi r^2 D_r$$

$\Rightarrow$ $$D_r = \frac{Q}{4\pi r^2}$$

or $$\overline{D} = \frac{Q}{4\pi r^2}\overline{a}_r$$

and $$\overline{E} = \frac{Q}{4\pi \in_0 r^2} a_r$$

Which is similar to the formula derived by using Coulomb's law

1.8.3 Applications of Gauss's Law - Infinite Line Charge

Let us consider that charge is distributed along Z-axis with the charge density ρ_L C/m. Since the charge distribution is along a line, a cylinder of length '*l*' can be assumed that surrounds the line charge distribution as shown in Fig.1.25. Hence it is better to consider cylindrical coordinate system to find either $\overline{E}$ or $\overline{D}$ at a point 'p' on the surface of the cylinder.

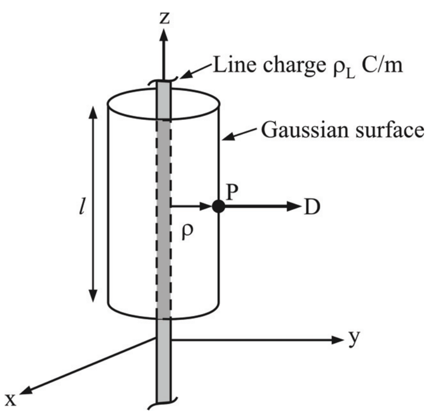


Fig. 1.25 Gaussian surface about an infinite line charge

Here $\overline{D}$ the electric flux density is perpendicular to the surface of the cylinder i.e., it will be in 'ρ' direction in cylindrical co-ordinate systems.

$\therefore \qquad \bar{D} = D_\rho \bar{a}_\rho$

The elemental surface ds lies on ϕ and Z axises

$\therefore \qquad d\bar{s} = \rho \; d\phi \; dz \; \bar{a}_\rho$

Flux flowing through the cylinder can be written as

$$\psi = \oint_S \bar{D} \cdot d\bar{s}$$

$$\psi = \int_{z=0}^{l} \int_{\phi=0}^{2\pi} D\rho \bar{a}_\rho . \rho d\phi dz \bar{a}_\rho$$

$$\psi = D_\rho \rho \int_{z=0}^{l} dz \int_{\phi=0}^{2\pi} d\phi$$

$$\psi = D_\rho 2\pi\rho l \qquad \text{.....(1.8.3)}$$

The charge enclosed by the cylinder is

$$Q_{enc} = \rho_L l \qquad \text{.....(1.8.4)}$$

According to Gauss's law

$$\psi = Q_{enc}$$

Substitute ψ and Q_{enc} from equations (1.8.3) and (1.8.4) in the above equation

$$\rho_L l = D_\rho 2\pi\rho l$$

$$\Rightarrow \qquad D_\rho = \frac{\rho_L}{2\pi\rho}$$

$$\bar{D} = \frac{\rho_L}{2\pi\rho} \bar{a}_\rho \quad \text{and}$$

$$\bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{\rho_L}{2\pi \epsilon_0 \rho} \bar{a}_\rho \qquad \text{.....(1.8.5)}$$

Which is similar to the formula derived by using Coulomb's law.

Problem 1.17

Given $\bar{D} = z\rho\cos^2\phi \bar{a}_z$ C/m^2. Calculate the charge density at (1, π/4, 3) and the total charge enclosed by the cylinder of radius 1m with $-2 \le z \le 2\ m$.

Solution

We know

$$\rho_v = \nabla \cdot \bar{D}$$

in cylindrical co-ordinate system the divergence can be written as

$$\rho_v = \frac{1}{\rho}\frac{\partial\left(\rho D_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial\left(D_\phi\right)}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\rho_v = \frac{\partial D_z}{\partial z} \quad \text{since } \bar{D} \text{ has only Z- component}$$

$$\rho_v = \rho \cos^2 \phi$$

$$\left(\rho_v\right)_{\left(1,\ \frac{\pi}{4},\ 3\right)} = (1)\cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2} \quad \text{C/m}^3$$

change enclosed = $Q_{enc} = \int_v \rho_v dv$ where $dv = \rho\, d\rho\, d\phi\, dz$

$$Q_{enc} = \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi}\int_{z=-2}^{2} \rho \cos^2 \phi \rho\, d\rho\, d\phi\, dz$$

$$= \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi} \rho^2 \cos^2 \phi (4)\, d\rho\, d\phi$$

$$= 4\int_{\rho=0}^{1} \rho^2 \left[\frac{1}{2}(2\pi) + \frac{1}{2}\sin 4\phi\right] d\rho$$

$$= 4\pi \int_{\rho=0}^{1} \rho^2 d\rho = 4\pi\left[\frac{\rho^3}{3}\right]_0^1 = \frac{4\pi}{3}\text{C}$$

Problem 1.18

If $\bar{D} = \left(2y^2 + z\right)\bar{a}_x + 4xy\,\bar{a}_y + x\,\bar{a}_z$ C/m^2. Find

(a) the volume charge density at (–1, 0, 3)

(b) the flux through the cube defined by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

(c) the total charge enclosed by the cube

Solution

According to Maxwell's I equation

$$\rho_v = \nabla \cdot \overline{D}$$

$$\rho_v = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$= 0 + 4x + 0$$

$$= 4x \text{ C/m}^3$$

(a) $(\rho_v)_{(-1,0,3)} = 4(-1) = -4 \text{ C/m}^2$

(b) & (c) $\psi = \int_v \rho_v dv = Q_{enc}$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} 4x \, dx \, dy \, dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} 4x(1) \, dx \, dy$$

$$= \int_{x=0}^{1} 4x(1) \, dx$$

$$= 4\left[\frac{x^2}{2}\right]_0^1 = \frac{4}{2} = 2C$$

Problem 1.19

Given the electric flux density $\overline{D} = 0.3r^2\overline{a}_r$ nC/m^2, in free space. Find

(a) $\overline{E}$ at point (2, 25°, 90°)

(b) the total charge within the sphere r = 3

(c) the total electric flux leaving the sphere r = 4

Solution

(a) Given $\overline{D} = 0.3r^2\overline{a}_r$ nC/m^2

$$\therefore \quad \overline{E} = \frac{\overline{D}}{\epsilon_0} = \frac{0.3r^2\overline{a}_r}{8.854 \times 10^{-12}}$$

$$\left(\bar{E}\right)_{\left(2,\ 25^{\circ},90^{\circ}\right)}=\frac{0.3(4)}{8.854\times10^{-12}}\bar{a}_r=1.355\times10^{11}\bar{a}_r\times10^{-9}=135.5\ \bar{a}_r\ \text{V/m}$$

(b) we know $\rho_v=\nabla\cdot\bar{D}$

$$=\frac{1}{r^2}\frac{\partial\left(r^2D_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\ D_\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}=\frac{1}{r^2}0.3(4)r^3\ \text{n}=1.2\text{r n}$$

Also known $Q=\int\limits_v\rho_v dv$ where $dv=r\sin\theta\, d\phi\, r\, d\theta\, dr$

$$=r^2\sin\theta\, d\theta\, d\phi\, dr$$

$$\therefore\quad Q=\int\limits_{r=0}^{3}\int\limits_{\theta=0}^{\pi}\int\limits_{\phi=0}^{2\pi}1.2r\,n\,r^2\sin\theta\, d\theta\, d\phi\, dr$$

$$=n\int\limits_{r=0}^{3}\int\limits_{\theta=0}^{\pi}1.2r^3\sin\theta(2\pi)d\theta\, dr$$

$$=2.4\pi n\int\limits_{r=0}^{3}r^3\left[-\cos\theta\right]_0^{\pi}dr$$

$$=2.4\pi n(2)\left[\frac{r^4}{4}\right]_0^3=305.4\ \text{nC}$$

(c) $Q=\int\limits_{r=0}^{4}\int\limits_{\theta=0}^{\pi}\int\limits_{\phi=0}^{2\pi}1.2\ nr^3\sin\theta\, d\theta\, d\phi\, dr$

Upon simplifying, we get

Q = 965.09 nC

1.8.4 Applications of Gauss's Law - Infinite Sheet of Charge

Consider an infinite sheet with surface charge density ρ_s C/m^2 is lying on XY plane as shown in the Fig.1.26. Since Electric flux density $\bar{D}$ is always normal to the surface, we need to find Electric flux density at any point on either side of the sheet. Since the charge distribution depends on X and Y axeses, rectangular coordinate system can be used to find $\bar{D}$ at any point on either side of the sheet.

Hence Consider a rectangular box that is cut symmetrically by the sheet as shown in the Fig.1.26. As $\bar{D}$ is perpendicular to the sheet it will have components only in the Z-direction i.e., components on X and Y-directions are zero. Let us find out $\bar{D}$ as

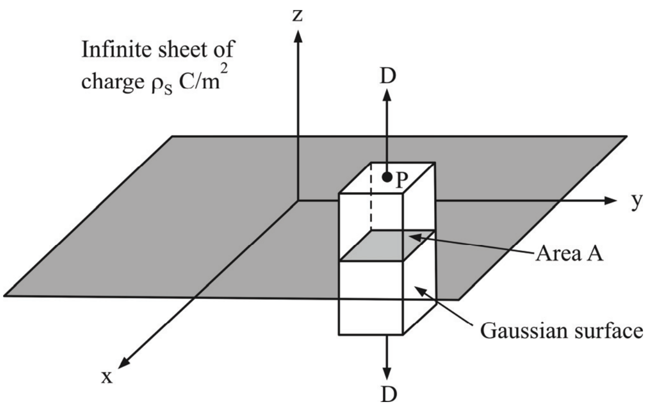


Fig.1.26 Gaussian surface about an infinite sheet of charge

Flux flowing through the rectangular box is

$$\psi = \oint_s \overline{D} \cdot \overline{ds}$$

Here $\overline{D} = D_z \overline{a}_z$ & $\overline{ds} = ds\overline{a}_z$

The flux due to bottom and top surfaces of rectangular box exists, but the flux due to the other surfaces of box is zero.

∴ above equation becomes

$$\psi = \oint_s D_z \overline{a}_z \cdot ds\overline{a}_z$$

$$\psi = D_z \left[\int_{top} ds + \int_{bottom} ds \right]$$

Assume that the area of the elemental surface as A, then

$$\psi = D_z [A + A]$$

$$\psi = 2AD_z$$

Charge enclosed by the rectangular box is

$$Q_{enc} = \int \rho_s ds$$

$$Q_{enc} = \rho_s \int ds$$

$$Q_{enc} = \rho_s A$$

According to Gauss's law

$$\psi = Q_{enc}$$

$$\rho_s A = 2AD_Z$$

$$D_z = \frac{\rho_s}{2}$$

$$\bar{D} = \frac{\rho_s}{2}\bar{a}_z$$

and $$\bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{\rho_s}{2\epsilon_0}\bar{a}_z \qquad .....(1.8.6)$$

which is similar to the formula derived by using Coulomb's law.

1.8.5 Applications of Gauss's Law - Uniformly Charged Sphere

Case I: (r < a)

Consider a sphere of radius 'a', which has uniform charge distribution with volume charge density ρ_v C/m^3 as shown in Fig.1.27. Since it is sphere, to find $\bar{D}$ at any point in side the sphere, consider a sphere of radius 'r' where r < a and is assumed as Gaussian surface. Hence spherical co-ordinate system can be used to find $\bar{D}$.

The charge enclosed by the sphere of radius 'r' is

$$Q_{enc} = \int_v \rho_v dv$$

$$Q_{enc} = \rho_v \int_v dv$$

$$= \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \rho_v \frac{4}{3}\pi r^3$$

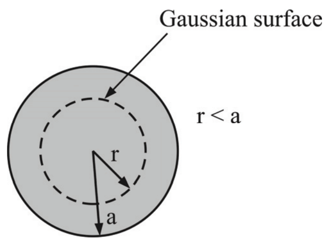


Fig. 1.27 Gaussian surface for uniformly charged sphere

The flux flowing through the spherical surface

$$\psi = \oint_s \bar{D}.\bar{ds}$$

As the flux density is normal to the surface it will have components only in 'r' direction.

$$= D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \, 4\pi r^2$$

According to Gauss's law charge enclosed = flux flowing through the surface i.e., $Q_{enc} = \psi$

$$\rho_v \frac{4}{3} \pi r^3 = D_r \, 4\pi r^2$$

$$D_r = \frac{\rho_v}{3} r$$

$$\bar{D} = \frac{\rho_v}{3} r \, \bar{a}_r$$

and $$\bar{E} = \frac{\bar{D}}{\in_0} = \frac{\rho_v}{3 \in_0} r \, \bar{a}_r \quad \text{.....(1.8.7)}$$

Case II (r > a)

To find the electric flux density out side the sphere of radius 'a', consider a sphere of radius 'r', which is treated as Gaussian surface as shown in Fig.1.28.

Charge enclosed by the sphere of radius 'r' is

$$Q_{enc} = \int_v \rho_v dv$$

$$Q_{enc} = \rho_v \int_v dv$$

$$= \rho_v \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{a} r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \rho_v \frac{4}{3} \pi a^3$$

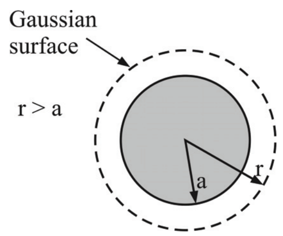


Fig. 1.28 Gaussian surface for uniformly charged sphere

Flux flowing through the surface

$$\psi = D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \, 4\pi r^2$$

$Q_{enc} = \psi$ according to Gauss's law

$$\rho_v \frac{4}{3} \pi a^3 = D_r \, 4\pi r^2$$

$$D_r = \frac{\rho_v}{3r^2} a^3$$

$$\bar{D} = \frac{\rho_v a^3}{3r^2} \bar{a}_r$$

and $$\bar{E} = \frac{\rho_v a^3}{3r^2 \in_0} \bar{a}_r$$

$$\bar{E} = \frac{\rho_v a^3 4\pi}{3r^2 \in_0 4\pi} \bar{a}_r$$

$$\bar{E} = \frac{Q}{4\pi \in_0 r^2} \bar{a}_r \quad \text{.....(1.8.8)}$$

which is similar to the formula derived by using Coulomb's law.

Problem 1.20

A charge distribution with spherical symmetry has density

$$\rho_v = \begin{cases} \rho_0 \dfrac{r}{R}, & 0 \le r \le R \\ 0, & r > R \end{cases}$$

Determine $\bar{E}$ everywhere

Solution:

Replace 'a' with 'R' in Fig.1.27, Then

Case I: Inside the sphere of radius 'R'

The charge enclosed by the sphere of radius 'r' is $Q_{enc} = \int_v \rho_v dv$

$$Q_{enc} = \int_v \rho_0 \frac{r}{R} dv$$

$$= \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} \frac{r^3}{R} \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{\rho_0}{R} \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{r=0}^{r} r^3 dr$$

$$= \frac{4\pi r^4 \rho_0}{4R}$$

$$Q_{enc} = \frac{\rho_0}{R} \pi r^4$$

The flux flowing through the spherical surface

$$\psi = \oint_s \bar{D}.\overline{ds}$$

As the flux density is normal to the surface it will have components only in 'r' direction.

$$= D_r \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\varphi$$

$$\psi = D_r \, 4\pi r^2$$

According to Gauss's law charge enclosed = flux flowing through the surface i.e., $Q_{enc} = \psi$

$$\frac{\rho_0}{R} \pi r^4 = D_r 4\pi r^2$$

$$D_r = \frac{\rho_0}{4R} r^2$$

$$\bar{D} = \frac{\rho_0}{4R} r^2 \, \bar{a}_r$$

and

$$\bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{\rho_0}{4R \epsilon_0} r^2 \, \bar{a}_r$$

Case II: Outside the sphere of radius 'R'

Charge enclosed by the sphere of radius 'r' is

$$Q_{enc} = \int_v \rho_v dv$$

$$Q_{enc} = \int_v \rho_0 \frac{r}{R} dv$$

$$= \frac{\rho_0}{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{R} r^3 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \rho_0 \pi R^3$$

Flux flowing through the surface

$$\psi = D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \, 4\pi r^2$$

$Q_{enc} = \psi$ according to Gauss's law

$$\rho_0 \, \pi R^3 = D_r \, 4\pi r^2$$

$$D_r = \frac{\rho_0 R^3}{4r^2}$$

$$\bar{D} = \frac{\rho_0 R^3}{4r^2} \bar{a}_r$$

and $$\bar{E} = \frac{\rho_0 R^3}{4r^2 \in_0} \bar{a}_r$$

Problem 1.21

A sphere of radius 'a' is filled with a uniform charge density of ρ_v C/m^3. Determine the electric field inside and outside the sphere.

Solution

The answer is as derived in section 1.8.5 case-I(inside the sphere) and case-II(outside the sphere).

Problem 1.22

A charge distribution in free space has ρ_v = 2r nC/m^3 for 0 < r < 10 m and '0' otherwise. Determine $\bar{E}$ at r = 2 m and r = 12 m

Solution

Replace 'a' with '10 m' in Fig.1.27, Then

At r = 2 m

The charge enclosed by the sphere of radius '2m' is $Q_{enc} = \int_v \rho_v dv$

$$Q_{enc} = \int_v 2rn dv$$

$$= 2n \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{2} r^3 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 32\pi \; nC$$

The flux flowing through the spherical surface

$$\psi = \oint_s \bar{D}.\bar{ds}$$

As the flux density is normal to the surface it will have components only in 'r' direction.

$$= D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \, sin\theta \, d\theta \, d\phi$$

$$= D_r \, 16\pi$$

According to Gauss's law charge enclosed = flux flowing through the surface i.e., $Q_{enc} = \psi$

$$32\pi \, n = D_r \, 16\pi$$

$$D_r = 2n$$

$$\bar{D} = 2n\,\bar{a}_r \text{ and}$$

$$\bar{E} = \frac{\bar{D}}{\epsilon_0} = 226\bar{a}_r \text{ V/m}$$

At $r = 12$ m

Charge enclosed by the sphere of radius '12 m' is

$$Q_{enc} = \int_v \rho_v dv$$

$$Q_{enc} = \int_v 2rndv$$

$$= 2n \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{10} r^3 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 20\pi \, \mu C$$

Flux flowing through the surface

$$\psi = D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \, sin\theta \, d\theta \, d\phi$$

$$= D_r \, 4\pi \, 12^2$$

$Q_{enc} = \psi$ according to Gauss's law

$$20\pi\,\mu = D_r\,4\pi\,12^2$$

$$D_r = 0.0347\mu$$

$$\bar{D} = 0.0347\mu\,\bar{a}_r \text{ and}$$

$$\bar{E} = 3.92\,\bar{a}_r\ \text{kV/m}$$

1.9 Electric Potential

To find electric field intensity $\bar{E}$, so far we have used Coulomb's law if the charge distribution is of any type and Gauss's law if the charge distribution has symmetry. Another method to find electric field intensity is by using electric potential which is a scalar. So obviously this method is easier when compared with the other two methods.

If we move a point charge from A to B in an electric field having electric field intensity $\bar{E}$ as shown in Fig.1.29.

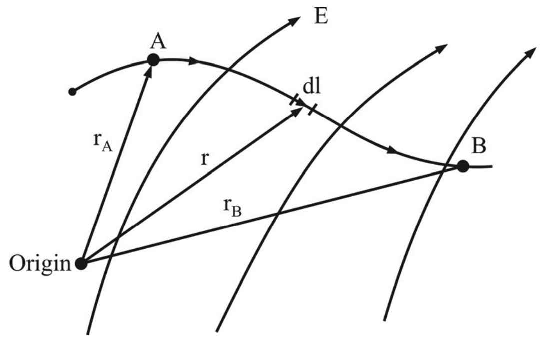


Fig. 1.29 Displacement of point charge in an electrostatic field

The elemental work done to move a point charge by an elemental distance dL is

$$dW = -\bar{F}\cdot d\bar{L}$$

The total work done in moving a point charge from A to B is

$$W = -\int_A^B \bar{F}\cdot d\bar{L}$$

-ve sign indicates work is being done by an external agent

We have $\bar{F} = Q\bar{E}$

then $$W = -\int_A^B Q\bar{E}\cdot d\bar{L}$$

$$W = -Q\int_A^B \overline{E} \cdot d\overline{L}$$

$$\Rightarrow \qquad \frac{W}{Q} = -\int_A^B \overline{E} \cdot d\overline{L}$$

which is work done per unit charge and it is also called potential difference V_{AB}.

We know that electric field intensity $\overline{E}$ due to a point charge is $\dfrac{Q}{4\pi \in_0 r^2}\overline{a}_r$

and elemental length $d\overline{L} = dr\overline{a}_r$

then $$V_{AB} = \int_{r_A}^{r_B} \frac{Q}{4\pi \in_0 r^2}\overline{a}_r \cdot dr\,\overline{a}_r$$

Where r_A and r_B are position vectors of point A and point B from origin

$$V_{AB} = \frac{Q}{4\pi \in_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$= \frac{Q}{4\pi \in_0 r_B} - \frac{Q}{4\pi \in_0 r_A} = V_B - V_A \qquad .....(1.9.1)$$

Where V_B and V_A are absolute potentials at point B and A respectively. From the above equation V_{AB} is the potential at B with reference to the potential at A.

If A is at ∞ then $V_A = 0$.

The above equation can be generalized for a potential (V) at any point having distance 'r' as

$$V = \frac{Q}{4\pi \in_0 r} \quad \text{(Here Q is located at origin)} \qquad .....(1.9.2)$$

If the point charge is placed at a distance r', then the electric potential at point 'r' can be written as

$$V = \frac{Q}{4\pi \in_0 |r - r'|} \qquad .....(1.9.3)$$

If we have 'n' number of point charges Q_1, Q_2, …,Q_n with position vectors r_1, r_2, …., r_n respectively, then the potential at 'r' is

$$V = \frac{Q_1}{4\pi \in_0 |r - r_1|} + \frac{Q_2}{4\pi \in_0 |r - r_2|} + ... + \frac{Q_n}{4\pi \in_0 |r - r_n|} \qquad .....(1.9.4)$$

For line charge distribution with charge density ρ_L, in the above equation Q can be replaced by $\int \rho_L dL$.

For surface charge distribution with charge density ρ_s, in equation (1.9.4), Q can be replaced by $\int \rho_S ds$.

Similarly Q can be replaced by $\int \rho_v dv$, For volume charge distribution with charge density ρ_v.

Problem 1.23

Two point charges – 4 μC and 5 μC are located at (2, –1, 3) and (0, 4, –2) respectively. Find the potential at (1, 0, 1). Assuming '0' potential at infinity.

Solution

$$V = \frac{Q_1}{4\pi \epsilon_0 |r - r_1|} + \frac{Q_2}{4\pi \epsilon_0 |r - r_2|}$$

$$V = \frac{-4\times 10^{-6}}{4\pi \epsilon_0 |(1,0,1) - (2,-1,3)|} + \frac{5\times 10^{-6}}{4\pi \epsilon_0 |(1,0,1) - (0,4,-2)|}$$

Simplifying, we get

$$V = -5.872 \text{ kV}$$

Problem: 1.24

A point charge 3 μC is located at the origin in addition to the two charges of previous problem. Find the potential at (–1, 5, 2). Assuming V(∞) = 0.

Solution:

$$r - r_1 = \sqrt{1+25+4} = 5.478$$

$$r - r_2 = \sqrt{9+36+1} = 6.782$$

$$r - r_3 = \sqrt{16+1+1} = 4.243$$

$$V = \left[\frac{3\times 10^3}{5.478} + \frac{-4\times 10^3}{6.782} + \frac{5\times 10^3}{4.243}\right]\times 9$$

$$= 10.23 \text{ kV}$$

Problem 1.25

A point charge of 5 nC is located at the origin if $V = 2$ V at (0, 6, –8) find

(a) the potential at A (–3, 2, 6)

(b) the potential at B (1, 5, 7)

(c) the potential difference V_{AB}

Solution

(a) $$V_A - V = \frac{Q}{4\pi \in_0}\left(\frac{1}{r_A} - \frac{1}{r}\right)$$

$$r_A = (-3, 2, 6) - (0, 0, 0) = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$r = (0, 6, -8) - (0, 0, 0) = \sqrt{0 + 6^2 + 8^2} = 10$$

$$V_A - 2 = \frac{5\times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}}\left(\frac{1}{7} - \frac{1}{10}\right)$$

V_A = 3.929 V

(b) $$V_B - V = \frac{Q}{4\pi \in_0}\left(\frac{1}{r_B} - \frac{1}{r}\right)$$

$$r_B = (1, 5, 7) - (0, 0, 0) = \sqrt{1 + 5^2 + 7^2} = \sqrt{75}$$

$$V_B - 2 = \frac{5\times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}}\left(\frac{1}{\sqrt{75}} - \frac{1}{10}\right)$$

V_B = 2.696 V.

(c) $V_{AB} = V_B - V_A = -1.233$ V

*Problem 1.26

A point of 5 nC is located at (–3, 4, 0), while line y = 1, z = 1 carries uniform charge 2 nC/m.

(a) If V = 0 V at O(0, 0, 0), find V at A(5, 0, 1).

(b) If V= 100 V at B(1, 2, 1), find V at C(–2, 5, 3).

(c) If V = – 5 V at O, find V_{BC}.

Solution:

Let the potential at any point be

$$V = V_Q + V_L$$

Where V_Q is potential due to point charge

i.e., $$V_Q = \frac{Q}{4\pi \in_0 r}$$

by neglecting constant of integration

and V_L is potential due to line charge distribution,

for infinite line, we have

$$\overline{E} = \frac{\rho_L}{2\pi \in_0 \rho}\overline{a}_\rho$$

$$\therefore \qquad V_L = -\int \overline{E}.d\overline{l} = -\int \frac{\rho_L}{2\pi \in_0 \rho}\overline{a}_\rho.d\rho\,\overline{a}_\rho$$

$$\therefore \qquad V_L = -\frac{\rho_L}{2\pi \in_0}\ln\rho$$

by neglecting constant of integration.

Here ρ is the perpendicular distance from the line y = 1, z = 1(which is parallel to the x-axis) to the field point.

Let the field point be (x, y, z), then

$$\rho = |(x, y, z) - (x, 1, 1)| = \sqrt{(y-1)^2 + (z-1)^2}$$

$$\therefore \qquad V = -\frac{\rho_L}{2\pi \in_0}\ln\rho + \frac{Q}{4\pi \in_0 r}$$

by neglecting constant of integration.

(a) $$\rho_O = |(0, 0, 0) - (0,1,1)| = \sqrt{2}$$

$$\rho_A = |(5,0,1) - (5,1,1)| = 1$$

$$r_O = |(0,0,0) - (-3,4,0)| = 5$$

$$r_A = |(5,0,1) - (-3,4,0)| = 9$$

$$\therefore \qquad V_O - V_A = -\frac{\rho_L}{2\pi \in_0}\ln\rho_O + \frac{\rho_L}{2\pi \in_0}\ln\rho_A + \frac{Q}{4\pi \in_0 r_O} - \frac{Q}{4\pi \in_0 r_A}$$

$$V_O - V_A = -\frac{\rho_L}{2\pi \in_0}\ln\frac{\rho_O}{\rho_A} + \frac{Q}{4\pi \in_0}\left[\frac{1}{r_O} - \frac{1}{r_A}\right]$$

$$0 - V_A = -\frac{2\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}}\ln\frac{\sqrt{2}}{1} + \frac{5\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}}\left[\frac{1}{5}-\frac{1}{9}\right]$$

$$-V_A = -36\ln\sqrt{2} + 45\left[\frac{1}{5}-\frac{1}{9}\right]$$

$$V_A = 36\ln\sqrt{2} - 4 = 8.477\ V$$

(b) $\rho_B = |(1,2,1)-(1,1,1)| = 1$

$$\rho_C = |(-2,5,3)-(-2,1,1)| = \sqrt{20}$$

$$r_B = |(1,2,1)-(-3,4,0)| = \sqrt{21}$$

$$r_C = |(-2,5,3)-(-3,4,0)| = \sqrt{11}$$

$$\therefore \quad V_C - V_B = -\frac{\rho_L}{2\pi\in_0}\ln\frac{\rho_C}{\rho_B} + \frac{Q}{4\pi\in_0}\left[\frac{1}{r_C}-\frac{1}{r_B}\right]$$

$$V_C - 100 = -36\ln\frac{\sqrt{21}}{1} + 45\left[\frac{1}{\sqrt{11}}-\frac{1}{\sqrt{21}}\right]$$

$$V_C - 100 = -51.052$$

$$V_C = 48.94\ \text{V}$$

(c) $V_{BC} = V_C - V_B = 48.94 - 100 = -51.052\ \text{V}$

1.10 Conservative and Non-Conservative Fields

1.10.1 Conservative Field

If the field is parallel to a straight line as shown in Fig. 1.30. Let $\overline{A}$ be a vector field. Choose a path P to Q as shown in Fig.1.30. $\overline{A}.d\overline{L}$ in moving from P to Q will be 'M' (scalar) and $\overline{A}.d\overline{L}$ in moving from Q to P is (–M).

$\therefore$ The $\oint \overline{A}.d\overline{L} = M - M = 0$. Chosen path may be of any shape, the contour line integral of $\overline{A}.d\overline{L}$ becomes '0'. The field whose contour line integral gives 'zero' is called conservative (or) irrotational field.

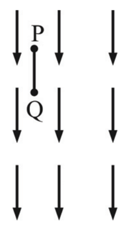


Fig. 1.30 Evaluation of conservative field

1.10.2 Non Conservative Field

In the conservative field, the filed vector is parallel to a straight line. Let us consider a field in circular fashion as shown in Fig.1.31(a). In this case $\overline{A}.d\overline{L}$ in moving from P to P along the field will not be 'zero' because $\overline{A}$ is always in the direction of $d\overline{L}$. These types of fields whose contour line integral of $\overline{A}.d\overline{L} \neq 0$ are called non conservative or rotational fields. The shape of the field need not be circular but it can be of any shape as shown in Fig. 1.31(b).

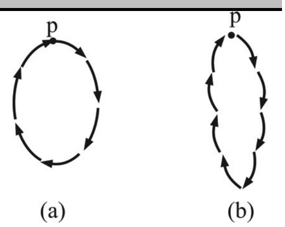


Fig. 1.31 Evaluation of non-conservative field

1.10.3 Concept of Curl

We know that in non-conservative fields as shown in Fig. 1.30 the contour line integral of $\overline{A}.d\overline{L}$ gives some finite value. This finite value is called circulation.

$\therefore$ circulation = $\oint \overline{A}.d\overline{L}$. This circulation depends upon the area chosen in the non conservative field. Let the area be ΔS. Then the ratio of $\oint \overline{A}.d\overline{L}$ to ΔS can be considered as one unit. As the field is normal to this unit we can write the above expression as $\dfrac{\oint \overline{A}.d\overline{L}}{\Delta S}\overline{a}_n$.

In general $\oint$ will be from point to point. This can be denoted by taking Limit $\Delta S \to 0$ which gives curl of vector $\overline{A}$ i.e.,

$$\nabla \times \overline{A} = \lim_{\Delta S \to 0} \frac{\oint \overline{A} \cdot d\overline{L}}{\Delta S}\overline{a}_n \qquad \text{.....(1.10.1)}$$

The curl of vector $\overline{A}$ gives circulation that exists on the chosen closed surface.

As $\nabla \times \overline{A}$ or curl of a vector $\overline{A}$ is a vector. It can be represented with three components in a rectangular co-ordinate system i.e., [curl $\overline{A}$]$_1$, [curl $\overline{A}$]$_2$, [curl $\overline{A}$]$_3$ along X, Y & Z axises with $\overline{a}_x, \overline{a}_y$ & $\overline{a}_z$ as unit vectors respectively.

$$\therefore \qquad \nabla\times \overline{A} = [\text{curl } \overline{A}]_1 + [\text{curl } \overline{A}]_2 + [\text{curl } \overline{A}]_3$$

To find [curl $\overline{A}$]$_1$ consider the elemental surface Δy and Δz which is normal to x-axis as shown in Fig. 1.32.

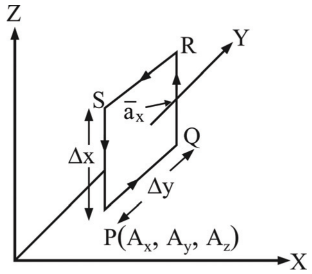


Fig. 1.32 Evaluation of curl

$$\therefore \text{ [curl } \overline{A}\,] = \lim_{\Delta y \Delta z \to 0} \frac{\oint \overline{A} \cdot d\overline{L}}{\Delta y \Delta z} a_x$$

Let the components of vector $\overline{A}$ at P be (A_x, A_y, A_z)

$\therefore$ The line integral from P to Q is $\int_P^Q \overline{A} \cdot d\overline{L}$.

$\because$ PQ is parallel to Y-axis, $\overline{A}$ can be taken as the Y component of $\overline{A}$ and the elemental length $d\overline{l}$ can be taken as Δy.

$\therefore$ The above integral becomes $A_y \Delta_y$. At Q we have moved a distance by Δy. To find line integral from Q to R consider the Z component at Q (because QR is parallel to Z-axis)

$\therefore$ The 'Z' component at Q is $A_z + \frac{\partial A_z}{\partial y} \Delta y$

$\therefore$ The line integral i.e., $\int_Q^R \overline{A} \cdot d\overline{L} = \left(A_z + \frac{\partial A_z}{\partial y} \Delta y \right) \Delta z$

At 'R' we have moved by a distance Δz. As RS line is parallel to Y-axis, consider the Y-component at 'R' as $A_y + \frac{\partial A_y}{\partial z} \Delta z$

$$\therefore \qquad \int_R^S \overline{A} \cdot d\overline{L} = \left(A_y + \frac{\partial A_y}{\partial z} \Delta z \right) (-\Delta y)$$

At 'S' to find $\int_S^P \overline{A} \cdot d\overline{L}$ consider the 'z' component at 'S' which is A_z

$$\therefore \qquad \int_S^P \overline{A} \cdot d\overline{L} = A_z (-\Delta z)$$

$$\therefore \qquad \oint \overline{A} \cdot d\overline{L} = \int_P^Q \overline{A} \cdot d\overline{L} + \int_Q^R \overline{A} \cdot d\overline{L} + \int_R^S \overline{A} \cdot d\overline{L} + \int_S^P \overline{A} \cdot d\overline{L}$$

$$= A_y \Delta y + A_z \Delta_z + \frac{\partial A_z}{\partial y} \Delta_y \Delta_z - A_y \Delta_y - \frac{\partial A_y}{\partial z} \Delta z \Delta y - A_z \Delta z$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \Delta z \Delta y$$

Substitute the above equation in [Curl $\overline{A}$]$_1$ equation

$$\therefore \qquad [\text{Curl } \bar{A}]_1 = \frac{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\Delta z \Delta y}{\Delta z \Delta y}\bar{a}_x = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\bar{a}_x$$

Similarly we can also construct equations for [Curl $\bar{A}$]$_2$ by considering the elemental surface on ZX plane which is perpendicular to Y-axis

$$\therefore \qquad [\text{Curl } \bar{A}]_2 = \bar{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

and for [Curl $\bar{A}$]$_3$ we have to consider the elemental surface on XY plane which is perpendicular to Z-axis

$$\therefore \qquad [\text{Curl } \bar{A}]_3 = \bar{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Curl $\bar{A}$ = [Curl $\bar{A}$]$_1$+ [Curl $\bar{A}$]$_2$+ [Curl $\bar{A}$]$_3$

$$\nabla \times \bar{A} = \bar{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \bar{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \bar{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Which can be written in matrix form as

Cartesian co-ordinate system:

$$\nabla \times \bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical co-ordinate system:

$$\nabla \times \bar{A} = \frac{1}{\rho}\begin{vmatrix} \bar{a}_\rho & \rho\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

Spherical co-ordinate system:

$$\nabla \times \bar{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

Stoke's Theorem

Stoke's theorem gives the relation between the line integral and surface integral as

$$\int_S \nabla \times \overline{A} \cdot d\overline{s} = \oint_L \overline{A} \cdot d\overline{L} \qquad \text{.....(1.10.2)}$$

where $\overline{A}$ is the field vector. According to above equation finding curl of a vector at every point in a chosen surface and adding all those values will be equal to the contour line integral of the boundary of the chosen surface.

Proof:

Let us consider a rotational field and choose a surface on it as shown in Fig.1.33.

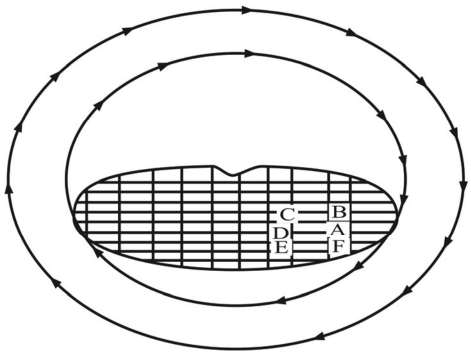


Fig. 1.33 Rotational field to explain Stoke's Theorem

We know that

$$\nabla \times \overline{A} = \lim_{\Delta s \to 0} \frac{\oint \overline{A} \cdot d\overline{L}}{\Delta S} \overline{a}_n$$

The above equation can be written as

$$\int_S \nabla \times \overline{A} \cdot d\overline{s} = \oint_L \overline{A} \cdot d\overline{L}$$

Which can be proved as

Choose a sub-surface Δs_1 (ABCDA). Then above equation becomes

$$\int_S \nabla \times \overline{A} \cdot d\overline{s_1} = \int_A^B \overline{A} \cdot d\overline{L} + \int_B^C \overline{A} \cdot d\overline{L} + \int_C^D \overline{A} \cdot d\overline{L} + \int_D^A \overline{A} \cdot d\overline{L}$$

choose one more sub-surface Δs_2 adjacent to Δs_1 which is (ADEFA)

$$\int_S \nabla \times \overline{A} \cdot d\overline{s}_2 = \int_A^D \overline{A} \cdot d\overline{L} + \int_D^E \overline{A} \cdot d\overline{L} + \int_E^F \overline{A} \cdot d\overline{L} + \int_F^A \overline{A} \cdot d\overline{L}$$

Let $\Delta s = \Delta s_1 + \Delta s_2$

$$\int_S \nabla \times \overline{A} \cdot d\overline{s} = \int_S \nabla \times \overline{A} \cdot d\overline{s}_1 + \int_S \nabla \times \overline{A} \cdot d\overline{s}_2$$

$$= \int_A^B \overline{A} \cdot d\overline{L} + \int_B^C \overline{A} \cdot d\overline{L} + \int_C^D \overline{A} \cdot d\overline{L} + \int_D^A \overline{A} \cdot d\overline{L} - \int_D^A \overline{A} \cdot d\overline{L} + \int_D^E \overline{A} \cdot d\overline{L} + \int_E^F \overline{A} \cdot d\overline{L} + \int_F^A \overline{A} \cdot d\overline{L}$$

$$= \int_A^B \overline{A} \cdot d\overline{L} + \int_B^C \overline{A} \cdot d\overline{L} + \int_C^D \overline{A} \cdot d\overline{L} + \int_D^E \overline{A} \cdot d\overline{L} + \int_E^F \overline{A} \cdot d\overline{L} + \int_F^A \overline{A} \cdot d\overline{L}$$

From the above equation by finding curl of a vector $\overline{A}$ at all the points in a chosen surface and adding up all the values will be equal to the contour line integral of the chosen boundary surface. Adding up all the curls is nothing but integrating the curl of a vector w.r.t. chosen surface.

$$\therefore \int_S \nabla \times \overline{A} \cdot d\overline{s} = \oint_L \overline{A} \cdot d\overline{L}$$

1.11 Relation Between $\overline{E}$ and V

We know that the potential difference between points A and B is $V_{AB} = -\int_A^B \overline{E} \cdot d\overline{L}$.

Similarly the potential difference from B to A is $V_{BA} = \int_A^B \overline{E} \cdot d\overline{L}$

$\therefore$ The total potential from moving A to B and back to A is

$$V_{AB} + V_{BA} = -\int \overline{E} \cdot d\overline{L} + \int_A^B \overline{E} \cdot d\overline{L} = 0 = \oint_L \overline{E} \cdot d\overline{L} \qquad \text{.....(1.11.1)}$$

The total work done in moving a point charge from A to B and back to A is ‘0’.

From equation (1.11.1) we can say that the electrostatic fields are conservative fields or irrotational fields.

According to Stoke’s theorem $\int_S \nabla \times \overline{E} \cdot d\overline{s} = \oint_L \overline{E} \cdot d\overline{L}$

$\therefore \qquad \int_S \nabla \times \overline{E} \cdot d\overline{s} = 0$ or $\nabla \times \overline{E} = 0$ .....(1.11.2)

Equation (1.11.1) is a Maxwell's second equation which is in integral form. Equation (1.11.2) is also a Maxwell's second equation which is in differential form

We know the potential difference $V = -\int \overline{E} \cdot d\overline{L}$

$$dv = -\overline{E} \cdot d\overline{L}$$

As $\overline{E}$ and $d\overline{L}$ are vectors they can be represented in rectangular co-ordinate system as

$$\overline{E} = E_x \overline{a}_x + E_y \overline{a}_y + E_z \overline{a}_z$$

$$d\overline{L} = dx\overline{a}_x + dy\overline{a}_y + dz\overline{a}_z$$

$\therefore \qquad dv = -\left(E_x dx + E_y dy + E_z dz\right)$ .....(1.11.3)

In calculus dv can be represented as

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \qquad \text{.....(1.11.4)}$$

from (1.11.3) & (1.11.4)

$$E_x = \frac{-\partial v}{\partial x}, \quad E_y = \frac{-\partial v}{\partial y} \text{ and } E_z = \frac{-\partial v}{\partial z}$$

$$\overline{E} = \frac{-\partial v}{\partial x}\overline{a}_x - \frac{\partial v}{\partial y}\overline{a}_y - \frac{\partial v}{\partial z}\overline{a}_z$$

$$\overline{E} = \frac{-\partial v}{\partial x}\overline{a}_x - \frac{\partial v}{\partial y}\overline{a}_y - \frac{\partial v}{\partial z}\overline{a}_z$$

$$\overline{E} = -\nabla V$$

Which is the relation between $\overline{E}$ and V.

Problem 1.27

Given the potential $V = \frac{10}{r^2}\sin\theta\cos\phi$

(a) Find the electric flux density $\overline{D}$ at $(2, \pi/2, 0)$

(b) Calculate the work done in moving a 10 mC charge from point A(1, 30°, 120°) to B(4, 90°, 60°)

Solution

(a) We have

$$\bar{E} = -\nabla V$$

Since V is given in spherical co-ordinate system, consider ∇V in spherical co-ordinate system

$$\therefore\ \bar{E} = -\frac{\partial v}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial v}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi}\bar{a}_\phi$$

$$= -\left(10\left(-2r^{-3}\right)\sin\theta\cos\phi\,\bar{a}_r + \frac{1}{r}\frac{10\cos\theta\cos\phi}{r^2}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{10\sin\theta(-\sin\phi)}{r^2}\bar{a}_\phi\right)$$

$$= -\left(10\left(-2r^{-3}\right)\sin\theta\cos\phi\,\bar{a}_r + \frac{1}{r}\frac{10\cos\theta\cos\phi}{r^2}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{10\sin\theta(-\sin\phi)}{r^2}\bar{a}_\phi\right)$$

$$= \left(\frac{20\sin\theta\cos\phi}{r^3}\bar{a}_r + \frac{-10\cos\theta\cos\phi}{r^3}\bar{a}_\theta + \frac{10\sin\phi}{r^3}\bar{a}_\phi\right)$$

$$= \frac{10}{r^3}\left(2\sin\theta\cos\phi\,\bar{a}_r - \cos\theta\cos\phi\,\bar{a}_\theta + \sin\phi\,\bar{a}_\phi\right)$$

$$\bar{D} = \bar{E}\in_0$$

$$= \frac{8.825\times10^{-11}}{r^3}\left[2\sin\theta\cos\phi\,\bar{a}_r - \cos\theta\cos\phi\,\bar{a}_\theta + \sin\phi\,\bar{a}_\phi\right]$$

$$= \frac{8.825\times10^{-11}}{r^3}\left[2.1.1\bar{a}_r - 0 + 0\right]$$

$$\bar{D}(2, \frac{\pi}{2}, 0) = 22.1\ \bar{a}_r\ \text{pC/m}^2$$

(b) Work done = $-Q\int_A^B \bar{E}\cdot d\bar{L} = -Q\left(-V_{AB}\right)$

$$= Q\,(V_B - V_A)$$

$$V_B = \frac{10}{16}1.\frac{1}{2} = 0.3125\ V$$

$$V_A = \frac{10}{1}\frac{1}{2}\left(-0.5\right) = -5\times0.5 = -2.5\ \text{V}$$

$V_B - V_A = 2.8125$ V

W = $10^{-3} \times 10 \times (V_B - V_A) = 28.125$ mJ

Problem 1.28

Given that $\bar{E} = \left(3x^2 + y\right)\bar{a}_x + x\bar{a}_y$ kV/m. Find the work done in moving a –2 μC charge from (0, 5, 0) to (2, –1, 0) by taking the path

(a) (0, 5, 0) →(2, 5, 0) → (2, –1, 0)

(b) $y = 5 - 3x$

Solution

(a) Line equation for (0, 5, 0) to (2, 5, 0) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\frac{x - 0}{0 - 2} = \frac{y - 5}{5 - 5} = \frac{z - 0}{0 - 0}$$

$y = 5$ $\quad$ $z = 0$

$dy = 0$ $\quad$ $dz = 0$

$$W_1 = -QK \int_{(0,\,5,\,0)}^{(2,5,0)} \left(\left(3x^2 + y\right)\bar{a}_x + x\bar{a}_y\right)\left(dx\,\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z\right)$$

$$= -QK \int_{(0,\,5,\,0)}^{(2,\,5,\,0)} \left(3x^2 + y\right)dx + xdy$$

$$= 2\times10^{-3} \int_{(0)}^{(2)} \left(3x^2 + 5\right)dx + 0$$

$$= 2\times10^{-3} \left(3\left(\frac{x^3}{3}\right)_0^2 + 5(2)\right)$$

$= 36$ mJ

Line equation for (2, 5, 0) to (2, –1, 0)

$Z = 0$ $\quad$ $dz = 0$

$$\frac{x - 2}{2 - 2} = \frac{y - 5}{5 + 1} = \frac{z - 0}{0 - 0}$$

$x = 2$ $\quad$ d$x = 0$

$$W_2 = -QK \int_{(2,\,5,\,0)}^{(2,\,-1,\,0)} \left(3x^2 + y\right)dx + xdy$$

$$W_2 = -QK \int_{5}^{-1} 2dy = -2QK\left(-1-5\right) = -24 \text{ mJ}$$

$W = W_1 + W_2 = 12$ mJ

(b) Line equation for (0, 5, 0) to (2, 5, 0) is $y = 5 - 3x$

$dy = -3dx$

$$W = -QK \int_{(0,\,5,\,0)}^{(2,\,-1,\,0)} \left(3x^2 + y\right)dx + xdy$$

$$W = 2\times10^{-3}\int_{0}^{2}\left(3x^2 + 5 - 3x\right)dx - 3x\,dx = 12 \text{ mJ}$$

1.12 Electric Dipole and Flux Lines

Electric dipole is formed by separating two point charges of equal magnitude but opposite in sign by a small distance.

Consider an electric dipole along Z-axis separated by a small distance ‘d’ as shown in Fig. 1.34.

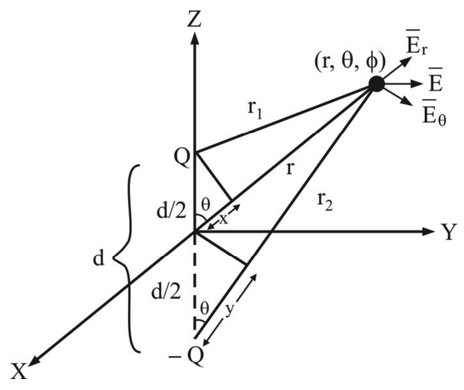


Fig. 1.34 Electric dipole to find potential

Let us find potential at $P(r,\theta,\phi)$ due to electric dipole. We know the potential at 'P' due to a point charge +Q is $V_Q = \dfrac{Q}{4\pi \in_0 r_1}$ and potential at 'P' due to –Q is $V_{-Q} = \dfrac{-Q}{4\pi \in_0 r_2}$

Potential at 'P' due to electric dipole is

$$V = \frac{Q}{4\pi \in_0 r_1} - \frac{Q}{4\pi \in_0 r_2}$$

$$= \frac{Q}{4\pi \in_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

from Fig.1.34 $\cos\theta = \dfrac{x}{d/2} \Rightarrow x = \dfrac{d}{2}\cos\theta$

$$r_1 = r - x$$

$$\therefore \quad r_1 = -\frac{d}{2}\cos\theta + r$$

$$\cos\theta = \frac{y}{d/2} \Rightarrow y = \frac{d}{2}\cos\theta$$

$$r = r_2 - y$$

$$\therefore \quad r_2 = r + \frac{d}{2}\cos\theta$$

$$\therefore \quad V = \frac{Q}{4\pi \in_0}\left(\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right)$$

$$= \frac{Q}{4\pi \in_0}\left(\frac{d\cos\theta}{r^2 - \left(\frac{d}{2}\cos\theta\right)^2}\right)$$

if r >>d

$$V = \frac{Q}{4\pi \in_0}\left(\frac{d\cos\theta}{r^2}\right) \qquad \text{.....(1.12.1)}$$

$$V \propto \frac{1}{r^2} \text{ due to dipole}$$

$\overline{d}.\overline{a}_r$ = d cosθ and here define electric dipole moment $\overline{p} = Q\overline{d}$ whose unit is C-m.

$$\therefore \qquad V = \frac{Q(\overline{d} \cdot \overline{a}_r)}{4\pi \epsilon_0 r^2} = \frac{(\overline{p} \cdot \overline{a}_r)}{4\pi \epsilon_0 r^2} = \frac{1}{4\pi \epsilon_0 r^2}\overline{p}.\frac{\overline{r}}{|\overline{r}|} \qquad \text{.....(1.12.2)}$$

If the electric dipole center is other than origin, let it be at r' then the above equation can be generalized as

$$V = \frac{1}{4\pi \epsilon_0 |r - r'|^2}\overline{p}.\frac{r - r'}{|r - r'|} = \frac{\overline{p}.(r - r')}{4\pi \epsilon_0 |r - r'|^3} \qquad \text{.....(1.12.3)}$$

The electric field due to dipole with center at the origin can be obtained as

$\overline{E} = -\nabla V$

Since V in equation (1.12.1) is in terms of r and θ consider ∇V in spherical co-ordinate system, then

$$\overline{E} = -\frac{\partial v}{\partial r}\overline{a}_r - \frac{1}{r}\frac{\partial v}{\partial \theta}\overline{a}_\theta$$

$$\overline{E} = \left(-(-2)r^{-3} \cos\theta\,\overline{a}_r - (-\sin\theta)\frac{1}{r}\frac{1}{r^2}\overline{a}_\theta\right)\frac{Qd}{4\pi \epsilon_0}$$

$$= \frac{Qd}{4\pi \epsilon_0}\left(2r^{-3} \cos\theta\,\overline{a}_r + \frac{\sin\theta}{r^3}\overline{a}_\theta\right)$$

$$= \frac{Qd}{4\pi \epsilon_0 r^3}\left(2\cos\theta\,\overline{a}_r + \sin\theta\,\overline{a}_\theta\right)$$

$$= \frac{p}{4\pi \epsilon_0 r^3}\left(2\cos\theta\,\overline{a}_r + \sin\theta\,\overline{a}_\theta\right) \qquad \text{.....(1.12.4)}$$

Problem 1.29

An electric dipole located at the origin in free space has a moment $\overline{p} = 3\overline{a}_x - 2\overline{a}_y + \overline{a}_z$ nCm

(a) Find V at P_A (2, 3, 4)

(b) Find V at $r = 2.5$, $\theta = 30^\circ$, $\phi = 40^\circ$

Solution

(a) We have

$$V = \frac{1}{4\pi \in_0 |r-r'|^2} \overline{p} \cdot \frac{r-r'}{|r-r'|}$$

$r' = (0, 0, 0)$

$$|r-r'| = \sqrt{4+9+16} = \sqrt{29}$$

$$V = 9\times 10^9 \frac{(3\overline{a}_x - 2\overline{a}_y + \overline{a}_z)\cdot(2\overline{a}_x + 3\overline{a}_y + 4\overline{a}_z)}{29\sqrt{29}} \times 10^{-9}$$

$$= \frac{9\times(4)}{(29)^{3/2}} = 0.235 \text{ V}$$

(b) $r = 2.5 \quad \theta = 30^{\circ} \quad \phi = 40^{\circ}$

$x = r \sin\phi \cos\theta = 0.958$

$y = r \sin\phi \sin\theta = 0.8035$

$z = r \cos\theta = 2.165$

upon simplifying we get

$V = 1.97$ V

Electric Flux Line

Electric flux line is an imaginary path or line drawn such that it's direction at any point is the direction of electric field intensity.

Equipotential Surface

Any surface which has same potential at all points is called as an equipotential surface.

Equipotential Line

The intersection line of equipotential surface with the plane is called as equipotential line. The work done to move a point charge from one point to other point along equipotential line is '0'.

The example for equipotential surface for a point charge is shown in Fig.1.35

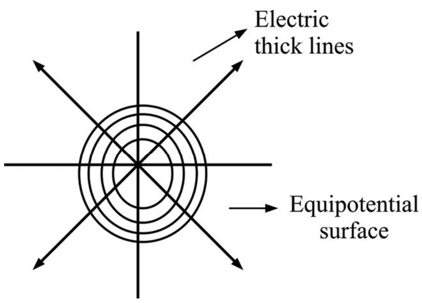


Fig. 1.35 Equipotential surface

Energy Density of Electrostatic Field

To find energy in the assembly of charges. Let us find the work required to assemble the charges. Consider a free surface and three point charges Q_1, Q_2 and Q_3 which are at infinity. The work required to move Q_4 from infinity to P_1 is $W_1 = 0$.

($\because$ initially the surface has no charge i.e., $\bar{\boldsymbol{E}} = 0 \therefore W_1 = -Q_1 \int \bar{E} \cdot Q\bar{L} = 0$)

The work required to move Q_2 from ∞ to P_2 which is shown in Fig.1.36 is $W_2 = Q_2 V_{21}$ where V_{21} is potential at P_2 due to Q_1. The work required to move Q_3 from ∞ to P_3 is $W_3 = Q_3(V_{32} + V_{31})$. Where V_{32} is potential at P_3 due to Q_2, V_{31} is potential at P_3 due to Q_1.

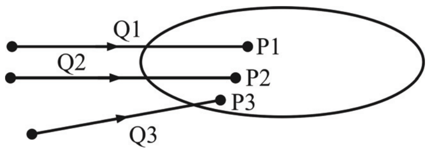


Fig. 1.36 Assembling of charges

$$\therefore \; W_E = W_1 + W_2 + W_3$$

$$= Q_2V_{21} + Q_3(V_{32} + V_{31}) \quad \text{.....(1.12.5)}$$

Suppose if we move initially Q_3 from ∞ to a free surface at P_3. The work required is $W_3 = 0$. Then work required to move Q_2 from ∞ to P_2 is $W_2 = Q_2 V_{23}$. Work required to move Q_1 from ∞ to P_1 is $W_1 = Q_1(V_{12} + V_{13})$

$$\therefore \; \text{Total work done } W_E = W_1 + W_2 + W_3$$

$$= 0 + Q_2V_{23} + Q_1(V_{12} + V_{13}) \quad \text{.....(1.12.6)}$$

Add (1.12.5) and (1.12.6)

$$2W_E = Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{32} + V_{31})$$

$$= Q_1V_1 + Q_2V_2 + Q_3V_3$$

$$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3) \quad \text{.....(1.12.7)}$$

Where V_1 is potential at P_1 due to Q_2 & Q_3, V_2 is potential at P_2 due to Q_1 & Q_3 and V_3 is potential at P_3 due to Q_1 & Q_2.

If we have 'n' number of charges the work required to bring them from ∞ to a surface which has initially zero charge is

$$W_E = \frac{1}{2}\sum_{k=1}^{n} Q_k V_k \qquad \text{.....(1.12.8)}$$

If the surface is having continuous charge distribution then the above equation becomes

$$W_E = \frac{1}{2}\int_L \rho_L V\, dL \text{ for line charge distribution} \qquad \text{.....(1.12.9)}$$

$$W_E = \frac{1}{2}\int_S \rho_s V\, dS \text{ for surface charge distribution} \qquad \text{.....(1.12.10)}$$

$$W_E = \frac{1}{2}\int_v \rho_V V\, dv \text{ for volume charge distribution} \qquad \text{.....(1.12.11)}$$

According to Maxwell's first equation $\rho_v = \nabla \cdot \bar{D}$

$$\therefore \qquad W_E = \frac{1}{2}\int_v \left(\nabla \cdot \bar{D}\right) V\, dv \qquad \text{.....(1.12.12)}$$

We know $\nabla \cdot \bar{A}V = \bar{A} \cdot \nabla V + V\left(\nabla \cdot \bar{A}\right)$ where $\bar{A}$ a general vector and V is a scalar

$$\therefore \qquad \left(\nabla \cdot \bar{A}\right)V = \nabla \cdot \bar{A}V - \bar{A} \cdot \nabla V$$

i.e.,

$$\left(\nabla \cdot \bar{D}\right)V = \nabla \cdot \bar{D}V - \bar{D} \cdot \nabla V$$

from (1.12.12)

$$W_E = \frac{1}{2}\int_v \left(\nabla \cdot \bar{D}V - \bar{D} \cdot \nabla V\right) dv$$

$$W_E = \frac{1}{2}\int_v \nabla \cdot \bar{D}V\, dv - \frac{1}{2}\int_v \bar{D} \cdot \nabla V\, dv$$

According to divergence theorem, first integral can be written as

$$W_E = \frac{1}{2}\int_S \bar{D}V . d\bar{S} - \frac{1}{2}\int_v \bar{D} \cdot \nabla V\, dv$$

For point charges the potential $V \propto \frac{1}{r}$, $\bar{E} \propto \frac{1}{r^2}$

For dipoles the potential $V \propto \frac{1}{r^2}, \bar{E} \propto \frac{1}{r^3}$

Surface ds $\propto$ r^2

If we consider the point charges the product of V and $\bar{E} \propto \frac{1}{r^3}$ and product of $\bar{D}V$ and $d\bar{S} \propto \frac{1}{r}$. For very large surface the first integral will become zero.

$$\therefore \qquad W_E = -\frac{1}{2}\int_v \bar{D}\cdot\nabla V\, dv$$

$$= -\frac{1}{2}\int_v \bar{D}.(-\bar{E})dv = \frac{1}{2}\int_v \bar{D}.\bar{E}\, dv$$

$\because \bar{D} = \epsilon_0 \bar{E}$

$$\text{Energy} = W_E = \frac{1}{2}\int_v \epsilon_0 \bar{E}\cdot\bar{E}\, dv \text{ Joules} \qquad \text{.....(1.12.13)}$$

$$\text{The energy density J/m}^3 \text{ is } \frac{dW_E}{dv} = \frac{1}{2}\epsilon_0 E^2 = w_E \text{ J/m}^3 \qquad \text{.....(1.12.14)}$$

Problem 1.30

Three point charges –1 nC, 4 nC and 3 nC are located at (0, 0, 0), (0, 0, 1) and (1, 0, 0) respectively. Find the energy in the system.

Solution

$W_E = W_1 + W_2 + W_3$

$= 0 + Q_2V_{21} + Q_3(V_{31}+V_{32})$

$$= Q_2.\frac{Q_1}{4\pi\epsilon_0|r_2 - r_1|} + \frac{Q_3}{4\pi\epsilon_0}\left[\frac{Q_1}{|r_3 - r_1|} + \frac{Q_2}{|r_3 - r_2|}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left(Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right)$$

$$= \frac{1}{4\pi.\frac{10^{-9}}{36\pi}}\left(-4 - 3 + \frac{12}{\sqrt{2}}\right).10^{-18}$$

$$= 9\left(\frac{12}{\sqrt{2}} - 7\right) \text{ nJ} = 13.37 \text{ nJ}$$

Problem 1.31

Point charges Q_1 = 1 nC, Q_2 = – 2 nC, Q_3 = 3 nC and Q_4 = – 4 nC are positioned one at a time and in that order at (0, 0, 0), (1, 0, 0), (0, 0, –1) and (0, 0, 1) respectively. Calculate the energy in the system after each charge is positioned.

Solution

Energy after Q_1 is positioned is $W_1 = 0$

$$W_2 = Q_2 V_{21} = Q_2 \cdot \frac{Q_1}{4\pi \in_0 |r_2 - r_1|} = \frac{-2 \times 1 \times 10^{-18}}{4\pi . \frac{10^{-9}}{36\pi} |(1,0,0) - (0,0,0)|} = -18 \text{ nJ}$$

Energy after Q_2 is positioned $W_2' = W_1 + W_2 = -18$ nJ

Energy after Q_3 is positioned

$$W_3' = W_2' + Q_3(V_{32} + V_{31})$$

$$= -18 nJ + \frac{3 \times 10^{-9}}{4\pi . \frac{10^{-9}}{36\pi}} \left[\frac{-2 \times 10^{-9}}{|(0,0,-1) - (1,0,0)|} + \frac{1 \times 10^{-9}}{|(0,0,-1) - (0,0,0)|} \right]$$

$$= -29.18 \text{ nJ}$$

Energy after Q_4 is positioned

$$W_4' = W_3' + Q_4(V_{43} + V_{42} + V_{41}) = -68.27 \text{ nJ}.$$

1.13 Convection and Conduction Currents

We know that materials are classified into conductors and non conductors based on conductivity σ (siemens/m or S/m). If σ > 1, the materials are called conductors and if σ < 1, the materials are called non conductors. The materials whose conductivity lies between these two materials are called semiconductors. Technically conductors and non conductors are called metals and insulators respectively. The basic difference between conductors and dielectrics (insulators) is: Conductors posses more number of free electrons to flow current through it, Where as dielectrics contain less number of free electrons to flow current through it.

If σ>>1, the conductors are called super conductors.

Current 'i' can be defined as charge flowing through a surface per unit time

$\therefore \qquad i = \frac{dQ}{dt}$

Current Density

The current Δi flowing through a surface Δs is denoted as $J_n = \Delta i/\Delta S$ A/m^2.

$$\Delta i = J_n\, \Delta S$$

If current density Jn is perpendicular to the surface ΔS

$$\Delta i = J_n\, \Delta S$$

If J_n is not perpendicular to ΔS, then $\Delta i = \bar{J} \cdot \Delta \bar{S}$

The total current flowing through surface is $I = \int_S \bar{J} \cdot d\bar{s}$.

Based on how the current I is produced, the current densities are classified in to (i) convection current density (ii) conduction current density and (iii) displacement current density.

Convection Current Density

Conductors are not involved for flowing current in case of convection current. Hence it will not satisfy ohm's law. The current flowing through an insulating material like liquid or vacuum is convection current. A beam of electrons through a vacuum tube is an example of convection current.

Consider a filament which is having volume charge density ρ_v as shown in Fig.1.37

Consider an elemental volume $\Delta V = \Delta S \Delta L$ and assume that the current is flowing in y-direction with velocity U_y.

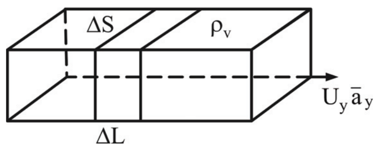


Fig.1.37 Current in a filament

We know that

$$\Delta Q = \rho_v\, \Delta V = \rho_v\, \Delta s\, \Delta l$$

Dividing with Δt

$$\frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta l}{\Delta t}$$

$$\Delta I = \rho_v \Delta S U_y \qquad \because \frac{\Delta Q}{\Delta t} = \Delta I \quad and \quad \frac{\Delta l}{\Delta t} = U_y$$

the current density $\qquad J = \frac{\Delta I}{\Delta S} = \rho_v U_y$

In general current density $\bar{J} = \rho_v \bar{U}$ .....(1.13.1)

which is convection current density and I is convection current.

Conduction Current Density

Conductors are involved in case of conduction current density. If we apply on electric field $\bar{E}$ to a conductor the force applied on electron which is having charge '– e' is

$$\bar{F} = -e\bar{E} \qquad \text{.....(1.13.2)}$$

If an electron having mass 'm' is moving with a drift velocity $\bar{U}$, according to Newton's law the average change in the momentum of electron is equal to the force applied on it.

Average change in momentum is $= \dfrac{m\bar{U}}{\tau}$ .....(1.13.3)

Equations (1.13.2) = (1.13.3)

i.e.,
$$\frac{m\bar{U}}{\tau} = -e\bar{E}$$

$$\bar{U} = \frac{-e\bar{E}\tau}{m}$$

Where τ = average time interval

m = mass of electron

If we have 'n' number of electrons in the considered conductor the volume charge density

$$\rho_v = -ne$$

We know that current density

$$\bar{J} = \rho_v \bar{U}$$

∴ Conduction current density $\bar{J} = -ne\dfrac{-e\bar{E}\tau}{m}$

$$\bar{J} = ne^2\bar{E}\frac{\tau}{m} \qquad \text{.....(1.13.4a)}$$

$$\bar{J} = \sigma\bar{E}$$

where

σ = conductivity of the conductor = $ne^2 \dfrac{\tau}{m}$ .....(1.13.4b)

Problem 1.32

If $\bar{J} = \frac{1}{r^3}\left(2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right)$ A/m^2. Calculate the current passing through

(a) Hemispherical shell of radius 20 cm.

(b) A spherical shell of radius 20 cm.

Solution

$$I = \int \bar{J}.\bar{ds}$$

Since it is sphere $\bar{ds} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$

(a) ϕ = 0 to 2π, θ = 0 to π/2 and r = 0.2 m for hemispherical shell

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \frac{1}{r^3}\left(2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right)\cdot r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r.a$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} 2\cos\theta\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{r}\int_{\theta=0}^{2\pi}\int_{\theta=0}^{\pi/2} \sin 2\theta\, d\theta\, d\phi$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} d\phi$$

$$= -\frac{1}{2r}(-1-1)(2\pi) = \frac{2\pi}{0.2} = 10\pi = 31.4A$$

(b) ϕ = 0 to 2π, θ = 0 to π and r = 0.2 m for spherical shell

$$I = \frac{1}{r}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} sin\, 2\theta\, d\theta\, d\phi$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi}\left[\frac{-cos\, 2\theta}{2}\right]_0^{\pi} d\phi$$

$$= -\frac{1}{2r}\int_{\phi=0}^{2\pi}[1-1]\, d\phi = 0A$$

Problem 1.33

For the current density $\overline{J} = 10z\sin^2\phi\,\overline{a}_\rho$ A/m^2. Find the current through the cylindrical surface $\rho = 2$, $1 \le z \le 5$ m.

Solution

Since it is cylinder $d\overline{s} = \rho\,d\phi\,dz\,\overline{a}_\rho$

We have

$$I = \int \overline{J} \cdot d\overline{s}$$

$$= \int_{z=1}^{5} \int_{\phi=0}^{2\pi} 10z\sin^2\phi\rho\,d\phi\,dz$$

$$= 10\rho \int_{z=1}^{5} z(1-\cos\phi)$$

$$= 754 \text{ A}$$

***Problem 1.34**

In a cylindrical conductor of radius 2 mm, the current density varies with distance from the axis according to $J = 10^3 e^{-400r}$ A/m^2. Find the total current I.

Solution

Since it is cylinder $d\overline{s} = \rho\,d\phi\,dz\,\overline{a}_\rho$

Here r = ρ = 0.02 m,

$$\therefore \quad \overline{J} = 10^3 e^{-400\rho}\ \overline{a}_\rho\ \ A/m^2$$

We know the total current $I = \int_s \overline{J}.d\overline{s}$

$$\therefore \quad I = \int_{\phi=0}^{2\pi} \int_{z=0}^{z} 10^3 e^{-400\rho} \rho d\phi\,dz$$

$$I = 2\pi z 10^3 e^{-400\rho} \rho$$

$$I = 4\pi z e^{-0.8} = z5.65 \text{ A}$$

Problem 1.35

If the current density $\overline{J} = \frac{1}{r^2}(\cos\theta \overline{a}_r + \sin\theta \overline{a}_\theta)\,\text{A/m}^2$, find the current passing through a sphere of radius 1.0 m.

Solution

We know the total current $I = \int_s \overline{J}.\overline{ds}$

Since it is spherical symmetry $\overline{ds} = r^2 \sin\theta d\theta d\phi\ \overline{a}_r$

$$\overline{J}.\overline{dS} = \frac{r^2}{r^2}\cos\theta \sin\theta\, d\phi\, d\theta$$

$$I = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \cos\theta \sin\theta\, d\phi\, d\theta$$

$$I = \pi\int_0^{\pi} \sin 2\theta\, d\theta$$

$$= \pi\left(\frac{-\cos 2\theta}{2}\right)_0^{\pi} = 0\ \text{A}$$

1.14 Polarization in Dielectrics

The basic difference between dielectrics and conductors is that dielectrics have less number of free electrons compared with the conducting material.

Consider a dielectric molecule with +Ve charge +Q (Nucleolus) and –Ve charge –Q (electron cloud) as shown in Fig. 1.38

To see the effect of electric field on dielectric materials consider the dielectric molecule as shown in Fig. 1.38. If we apply electric field $\overline{E}$ on to dielectric material, the force on positive charge is $\overline{F}_+ = Q\overline{E}$ which is along the direction of electric field $\overline{E}$ and the force on negitive charges is $\overline{E}$ = –Q $\overline{E}$ which is in opposite direction to $\overline{E}$.

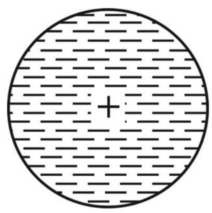


Fig. 1.38 Electron cloud

After applying electric field $\overline{E}$, charge is displaced as shown in Fig.1.39. The charge displacement is equal to sum of the original charge distribution and a dipole with dipole moment $\left(\overline{p} = Q\overline{d}\right)$ as shown in Fig.1.39.

After applying electric field, basically we get dipoles and hence the dielectric element is said to be polarized such dielectric material is said to be nonpolar. Examples are hydrogen, oxygen, nitrogen and the rare gases. Other types of molecules such as water, sulfur dioxide and hydrochloric acid have built-in permanent dipoles that are randomly oriented.

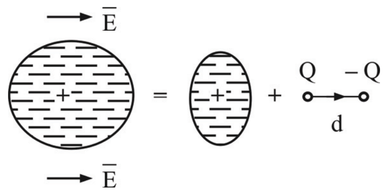


Fig. 1.39 Charge displacement after applying $\bar{E}$

Polarization

Creation of dipoles by applying electric field to the dielectric material is called polarization. Suppose 'N' numbers of dipoles are formed within 'ΔV' volume then the total number of dipole moments can be written as

$$= Q_1\bar{d}_1 + Q_2\bar{d}_2 + \ldots + Q_n\bar{d}_n = \sum_{k=1}^{N} Q_k\bar{d}_k$$

Polarization is defined as dipole moment/unit volume of the dielectric whose unit is (C/m^2)

$$\therefore \text{ Polarization} \qquad \bar{P} = \lim_{\Delta V \to 0} \frac{\sum_{k=1}^{N} Q_k\bar{d}_k}{\Delta V} \text{ C/m}^2 \qquad \ldots..(1.14.1)$$

Polarized(bounded) surface charge density $\rho_{ps} = \bar{P}\cdot\bar{a}_n$ and polarized (bounded) volume charge density $\rho_{pv} = -\nabla\cdot\bar{P}$

Consider a volume which has dielectric material with volume charge density ρ_v. Then the total volume charge density $\rho_T = \rho_v + \rho_{pv} = \nabla\cdot\bar{D}$

$$\rho_v + \rho_{pv} = \nabla\cdot\in_0 \bar{E}$$

$$\Rightarrow \qquad \rho_v = \nabla\cdot\in_0 \bar{E} - \rho_{pv}$$

$$\rho_v = \nabla\cdot\in_0 \bar{E} + \nabla\cdot\bar{P} \qquad \because \rho_{pv} = -\nabla\cdot\bar{P}$$

$$\rho_v = \nabla\cdot\left(\in_0 \bar{E} + \bar{P}\right)$$

$$\rho_v = \nabla\cdot\bar{D}$$

$$\text{where} \qquad \bar{D} = \in_0 \bar{E} + \bar{P} \qquad \ldots..(1.14.2)$$

The electric flux density $\bar{D}$ in free space is $\in_0 \bar{E}$ i.e., $\bar{P} = 0$ in free space.

From the above equation we can say that $\bar{D}$ is getting increased by $\bar{P}$ in dielectric materials.

From the discussion on polarization $\overline{P}$ is directly related with electric field $\overline{E}$

$$\therefore \qquad \overline{P} = X_E \in_0 \overline{E} \qquad \text{.....(1.14.3)}$$

Where X_E is the electric susceptibility. The value of parameter X_E gives how susceptible the given dielectric material to the applied electric field.

Dielectric constant and strength:

Substitute equation (1.14.2) in equation (1.14.1)

$$\overline{D} = \in_0 \overline{E} + X_E \in_0 \overline{E}$$

$$= \in_0 (1 + X_E)\overline{E}$$

$$= \in_0 \in_r \overline{E}$$

$$= \in \overline{E}$$

where $\quad \in = \in_0 . \in_r \quad \in_r = \dfrac{\in}{\in_0} = 1 + X_E$

Where $\in$ is the permittivity of dielectric material and $\in_0$ is the permittivity of free space and $\in_r$ is the dielectric constant or relative permittivity. The dielectric constant $\in_r$ can be defined as the ratio of $\in$ to $\in_0$.

If electric field strength is more such that it pulls the electrons from the outer shells of dielectric molecules, then the dielectric material becomes conducting material and we can say dielectric material has been broken.

$\therefore$ Dielectric strength can be defined as the maximum electric field with which dielectric material can tolerate or withstand.

1.15 Linear, Isotropic and Homogeneous Dielectrics

Dielectric materials can be classified into

(i) linear dielectrics

(ii) homogeneous dielectrics

(iii) isotropic dielectrics.

Linear Dielectrics: If $\in$ does not change with electric field then we can say the dielectric as linear dielectric.

Homogeneous Dielectrics: If $\in$ does not change from point to point then we can say the dielectric as homogeneous dielectric.

Isotropic dielectrics: If $\in$ does not change with the direction then we can say the dielectric as isotropic dielectric.

Similarly conducting materials are classified as

If 'σ' is independent of $\bar{E}$ then the conducting material is linear conducting material.

If 'σ' is independent of direction then the conducting material is isotropic conductor.

If 'σ' does not change from point to point then the conducting material is homogeneous conductor.

1.16 Continuity Equation and Relaxation Time

1.16.1 Continuity Equation

According to conservation of energy the rate of decrease of charge within a volume is equal to the net outward current flowing through a closed surface

$$\therefore \qquad I_{out} = \oint_S \bar{J} \cdot d\bar{s} = -\frac{dQ}{dt}$$

According to divergence theorem $\oint_S \bar{J} \cdot d\bar{s} = \int_v \nabla \cdot \bar{J}\, dv$ .....(1.16.1a)

$-\frac{dQ}{dt}$ can be written as $-\frac{dQ}{dt} = \frac{-d}{dt}\left[\int \rho_v dv\right]$

$$= -\int_V \left(\frac{\partial}{\partial t}\rho_v\right) dv \qquad \text{.....(1.16.1b)}$$

equations(1.16.1a) = (1.16.1b)

$$\int_v \nabla \cdot \bar{J} dv = -\int_v \left(\frac{\partial}{\partial t}\rho_v\right) dv$$

$$\therefore \qquad \nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \qquad \text{.....(1.16.1c)}$$

which is the continuity current equation.

The left side of the equation is the divergence of the Electric Current Density $(\bar{J})$. This is a measure of whether current is flowing into a volume (i.e., the divergence of $\bar{J}$ is positive if more current leaves the volume than enters).

Recall that current is the flow of electric charge. So if the divergence of $\bar{J}$ is positive, then more charge is exiting than entering the specified volume. If charge is exiting, then the amount of charge within the volume must be decreasing. This is exactly what the right side is a measure of how much electric charge is accumulating or leaving in a volume. Hence, the continuity equation is about continuity - if there is a net electric current is flowing out of a region, then the charge in that region must be decreasing. If there is more electric current flowing into a given volume than exiting, then the amount of electric charge must be increasing.

1.16.2 Relaxation Time

To derive the equation for relaxation time,

consider Maxwell's first equation i.e.,

$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \cdot \in \bar{E} = \rho_v$$

$$\nabla \cdot \bar{E} = \frac{\rho_v}{\in} \quad \text{.....(1.16.2)}$$

Consider the conduction current equation (point form of ohm's law)

$$\bar{J} = \sigma \bar{E} \quad \text{.....(1.16.3)}$$

From (1.16.2) $\nabla \cdot \sigma \bar{E} = \sigma \frac{\rho_v}{\in}$

$$\nabla \cdot \bar{J} = \sigma \frac{\rho_v}{\in} \quad \text{from (1.16.3)}$$

$$\frac{-\partial \rho_v}{\partial t} = \sigma \frac{\rho_v}{\in} \quad \text{from continuity equation}$$

$$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\in} \partial t$$

on integrating

$$\ln \rho_v = -\frac{\sigma}{\in} t + \ln \rho_{v0}$$

$$\frac{\rho_v}{\rho_{v0}} = e^{\frac{-\sigma}{\in} t} = e^{-t/(\in/\sigma)} \quad \text{.....(1.16.4)}$$

ρ_{v0} = initial volume change density

$$\frac{\in}{\sigma} = T_r$$

Which is relaxation time or rearrangement time.

Let us consider the effect of inserting the charge in the interior point of the material (Material can be conductor or dielectric).

Due to the insertion of charge in the interior point of the material, the volume charge density decreases exponentially.

Relaxation time can be defined as the time it takes a charge placed within an interior point of material to drop to e^{-1} = 36.8% of its initial value.

Relaxation time is very short for good conductors and high for good dielectrics. When we place a charge within a conductor within a short period charge disappears and it appears on the surface of conductor. Similarly when we place a charge within a dielectric material the charge remains there for a longer time.

1.17 Poisson's and Laplace's Equations

We can find $\bar{E}$ or $\bar{D}$ by using Coloumb's law or Gauss's law, (if the distribution is symmetry) if the charge distribution is known. We can also find out $\bar{E}$ or $\bar{D}$, if the potential difference is known. But in practical situation charge distribution and potential difference may not be given, in such cases either charge or potential is known only at boundary. Such type of situations or problems can be tackled either by using Poisson's equation or Laplace's equation.

We know Maxwell's first equation $\nabla \cdot \bar{D} = \rho_V$

Substitute $\bar{D} = \in \bar{E}$ in the above equation

$$\nabla \cdot \in \bar{E} = \rho_V$$

we know

$$\bar{E} = -\nabla V$$

$$\nabla \cdot \left(- \in \nabla V\right) = \rho_V \qquad \text{.....(1.17.1a)}$$

which is the Poisson's equation for in-homogeneous medium.

For charge free medium $\rho_v = 0$

$$\nabla \cdot \left(- \in \nabla V\right) = 0 \qquad \text{.....(1.17.1b)}$$

which is the Laplace's equation for in-homogeneous charge free medium.

For homogeneous medium since $\in$ is constant

$$\nabla^2 V = \frac{-\rho_v}{\in} \qquad \text{.....(1.17.2)}$$

which is the Poisson's equation for homogeneous medium.

For charge free region $\rho_v = 0$

$$\nabla^2 V = 0 \qquad \text{.....(1.17.3)}$$

which is the Laplace's equation for homogeneous charge free medium.

We know

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{.....(1.17.4)}$$

Which is Laplace's equation in rectangular co-ordinate system.

where ∇^2 is Laplacian operator

In cylindrical co-ordinate system is

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{.....(1.17.5)}$$

In spherical co-ordinate system

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \qquad \text{.....(1.17.6)}$$

Problem 1.36

Write Laplace's equation in rectangular co-ordinates for two parallel planes of infinite extent in the X and Y directions and separated by a distance 'd' in the Z-direction. Determine the potential distribution and electric field strength in the region between the planes.

Solution

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

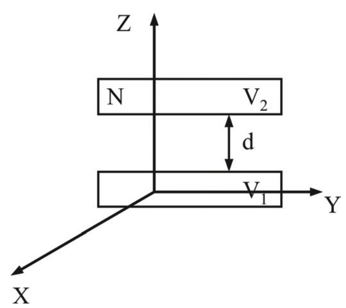


Fig. 1.40

since the potential is constant in X and Y directions

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial V}{\partial z} = A$$

$V = Az + B$

At $Z = 0$ $V = V_1$

$V_1 = 0 + B$

At $Z = d$ $V = V_2$

$V_2 = Ad + B$

$V_2 = Ad + V_1$

$$A = \frac{V_2 - V_1}{d}$$

The Potential distribution is $V = \frac{V_2 - V_1}{d} z + V_1$

The Electric field strength is

$$\overline{E} = -\nabla V = -\frac{\partial V}{\partial z}\overline{a}_z = -\frac{V_2 - V_1}{d}\overline{a}_z = \frac{V_1 - V_2}{d}\overline{a}_z$$

1.18 Parallel Plate Capacitor, Coaxial Capacitor, Spherical Capacitor

Capacitor may be obtained by separarting two conductors in some medium, which are having charges equal in magnitude but opposite in sign, such that the flux leaving from one surface of the conductor, terminates at the other conductor. Medium can be either free space or dielectric . Generally these conductors are called plates.

Let us consider two conductors with + Q and – Q charges and are connected to a voltage or potential difference 'V' as shown in the Fig.1.41.

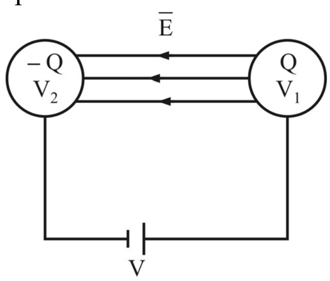


Fig. 1.41 Two conductors connected to V

The potential difference 'V' can be written in terms of $\overline{E}$ as potential difference $V = V_1 - V_2 = -\int_1^2 \overline{E} \cdot d\overline{L}$

The parameter of the capacitor i.e., 'capacitance' is defined as the ratio of charge on one of the conductors to the potential difference between two conductors.

$$C = \frac{Q}{V} = \frac{\oint_S \overline{D} \cdot d\overline{s}}{\int \overline{E} \cdot d\overline{L}} = \frac{\oint_S \in \overline{E} \cdot d\overline{s}}{\int \overline{E} \cdot d\overline{L}} \quad .....(1.18.1)$$

1.18.1 Parallel Plate Capacitor

Consider two conductors whose area as 'A' and are separated by a distance 'd' as shown in Fig.1.42.

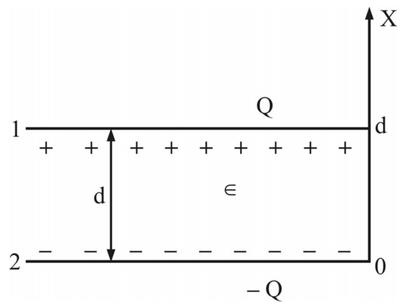


Fig. 1.42 Parallel plate capacitor

We know the electric field intensity $\overline{E}$ between parallel plate capacitors in free space as $\overline{E} = \frac{\rho_s}{\in_0} \overline{a}_n$

But from the Fig. 1.41 $\overline{E} = \frac{\rho_s}{\in}(-\overline{a}_x)$

$\because \overline{E}$ will be in opposite direction of x-axis

$$Q = \rho_s . A \Rightarrow \rho_s = \frac{Q}{A}$$

Where A = area of conductor.

$$\therefore \quad \overline{E} = -\frac{Q}{A \in} \overline{a}_x$$

We know the potential difference between two conductors which are separated by a distance 'd' as

$$V = -\int_0^d \overline{E} . d\overline{L}$$

where $d\overline{L} = dx\,\overline{a}_x$

$$V = -\int_0^d \frac{-Q}{A \in} \overline{a}_x . dx\,\overline{a}_x$$

$$V = \int_0^d \frac{Q}{A \in} dx = \frac{Qd}{A \in}$$

$$\therefore \quad C = \frac{Q}{V} = \frac{A \in}{d} \quad .....(1.18.2)$$

Energy stored in the parallel plate capacitor is

$$W_E = \frac{1}{2}\int_v \in \bar{E}\cdot\bar{E}\,dv$$

$$W_E = \frac{1}{2}\int_v \in \frac{\rho_s}{\in}\bar{a}_x \cdot \frac{\rho_s}{\in}\bar{a}_x dv$$

$$W_E = \frac{1}{2}\int_v \in \frac{\rho_s^2}{\in^2} dv$$

$$W_E = \frac{1}{2}\frac{\rho_s^2}{\in}\int_V dv = \frac{1}{2}\frac{\rho_s^2}{\in}\left(A\times d\right) = \frac{\rho_s^2 Ad}{2\in}$$

Replace ρ_s *by* $\frac{Q}{A}$

$$W_E = \frac{1}{2}\frac{Q^2}{A^2}\frac{Ad}{\in} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}VQ \qquad \text{.....(1.18.3)}$$

*Problem 1.37

Calculate the capacitance of a parallel plate capacitor with a dielectric, mica filled between plates. $\in_r$ of mica is 6. The plates of the capacitor are square in shape with 0.254 cm side. Separation between the two plates is 0.254 cm.

Solution

We have $C = \frac{\in A}{d}$

Here $\in = \in_0 \in_r = 8.854\times10^{-12}\times 6$

$$C = \frac{8.854\times10^{-12}\times6\times0.254\times0.254\times10^{-4}}{0.254\times10^{-2}} = 0.1349 \text{ pF}$$

*Problem 1.38

A parallel plate capacitance has 500 mm side plates of square shape separated by 10 mm distance. A sulphur slab of 6 mm thickness with $\in_r = 4$ is kept on the lower plate find the capacitance of the set-up. If a voltage of 100 volts is applied across the capacitor, calculate the voltages at both the regions of the capacitor between the plates.

Solution

Given

Area of parallel plates, A = 500 mm × 500 mm = 500 × 500 × 10^{-6} m^2.

Distance of separation d = 10 mm = 10×10^{-3} m.

Thickness of sulphur slab d_2 = 6 mm = 6×10^{-3} m.

Relative permittivity of sulphur slab $\in_r = 4$.

Voltage applied across the capacitor V = 100 V.

Here the capacitor has two dielectric media,

One medium is the sulphur slab of thickness (d_2)6 mm,

since the distance between the plates(d) is 10 mm

The remaining distance is air $d_1 = d - d_2$ = 4 mm.

∴ The other dielectric medium is air with thickness (d_1) 4 mm.

The capacitance of the parallel plate capacitor with two dielectric media is

$$C = \frac{\in_0 A}{\left(\frac{d_1}{\in_{r1}} + \frac{d_2}{\in_{r2}}\right)} \text{ F}$$

Here $\in_{r1}$ (air) = 1, $\in_{r2} = \in_r = 4$

$$C = \frac{8.854 \times 10^{-12} \times 500 \times 500 \times 10^{-6}}{\left(\frac{4 \times 10^{-3}}{1} + \frac{6 \times 10^{-3}}{4}\right)} = 0.402 \text{ nF}$$

The charge Q = CV = $0.402 \times 10^{-9} \times 100 = 4.02 \times 10^{-8}$ C

The value of capacitance (C_1) in delectric-1 i.e., air is

$$C_1 = \frac{\in_0 A}{d_1} = \frac{8.854 \times 10^{-12} \times 500 \times 500 \times 10^{-6}}{4 \times 10^{-3}} = 0.55 \text{ nF}$$

Similarly, The value of capacitance (C_2) in delectric-2 i.e., sulphur is

$$C_2 = \frac{\in A}{d_2} = \frac{4 \times 8.854 \times 10^{-12} \times 500 \times 500 \times 10^{-6}}{6 \times 10^{-3}} = 1.48 \text{ nF}$$

We have $V = V_1 + V_2$

Where V_1 is the voltage at the region of the capacitor plate near dielectric-1 i.e., air.

and V_2 is the voltage at the region of the capacitor plate near dielectric-2 i.e., sulphur.

$$V_1 = \frac{Q_1}{C_1} = \frac{Q}{C_1} = \frac{4.02 \times 10^{-8}}{0.55 \times 10^{-9}} = 73.1 \text{ V}$$

∴ $V_2 = 100 - 73.1 = 26.9$ V

1.18.2 Co-axial Capacitor

Consider two co-axial cables or co-axial cylinders of length 'L' where inner cylinder radius is 'a' and outer cylinder radius is 'b' as shown in Fig.1.43. The space between two cylinders is filled up with a homogeneous dielectric material with permittivity $\in$. Assume the charge on inner cylinder as Q and on the outer cylinder as –Q.

we have charge enclosed by the cylinder as

$$Q = \oint \overline{D} \cdot \overline{ds} \quad \text{where } \overline{D} = D_\rho \, \overline{a}_\rho \text{ and } \overline{ds} = \rho \, d\phi \, dz \, \overline{a}_\rho$$

$$\therefore \quad Q = D_\rho \rho \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{L} dz = 2\pi D_\rho \rho L = 2\pi \in E_\rho \rho L$$

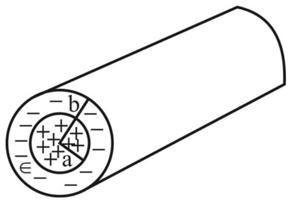


Fig. 1.43 Co-axial capacitor

i.e., $$E_\rho = \frac{Q}{\in 2\pi \rho L} \Rightarrow \overline{E} = \frac{Q}{2\pi \in \rho L} \overline{a}_\rho$$

To find the capacitance of co-axial capacitor. We need to find the potential difference between the two cylinders.

$$\therefore \quad V = -\int_b^a \overline{E}.\overline{dl} \quad \text{where } \overline{dl} = d\rho \, \overline{a}_\rho$$

$$V = -\int_b^a \overline{E}.d\rho \, \overline{a}_\rho$$

$$= -\int_b^a \frac{Q}{2\pi \in \rho L} d\rho$$

$$V = \frac{Q}{2\pi \in L} \ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{V} = \frac{2\pi \in L}{\ln\left(b/a\right)} \qquad \text{.....(1.18.4)}$$

Which is the expression for Coaxial capacitance.

1.18.3 Spherical Capacitor

Consider two spheres i.e., inner sphere of radius 'a' and outer sphere of radius 'b' which are separated by a dielectric medium with permittivity $\in$ as shown in Fig.1.44. The charge on the inner sphere is +Q and on the outer sphere is –Q.

We have charge enclosed by the sphere as

$$Q = \oint_S \bar{D} . d\bar{s}$$

where $\bar{D} = D_r \bar{a}_r$;

$$d\bar{s} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$Q = \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} r^2 \sin\theta \, D_r \, d\theta$$

$$D_r = \frac{Q}{4\pi r^2}$$

$$E_r = \frac{Q}{4\pi \in r^2}$$

$$\bar{E} = \frac{Q}{4\pi \in r^2} \bar{a}_r$$

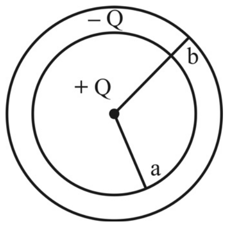


Fig. 1.44 Spherical capacitor

To find the capacitance of spherical capacitor. We need to find the potential difference between the two spheres.

$\therefore$ $$V = -\int_b^a \bar{E} \cdot d\bar{l}$$

where $$d\bar{l} = dr \, \bar{a}_r$$

$$V = \frac{-Q}{4\pi \in} \int_b^a \frac{1}{r^2} dr$$

$$= \frac{Q}{4\pi \in} \left[\frac{1}{a} - \frac{1}{b} \right]$$

$$C = \frac{Q}{V} = \frac{4\pi \in}{\left(\frac{1}{a} - \frac{1}{b} \right)} \qquad .....(1.18.5)$$

Which is the expression for Spherical capacitance.

Review Questions and Answers

1. State stokes theorem.

Ans. The line integral of a vector around a closed path is equal to the surface integral of the normal component of its curl over any surface bounded by the path.

$$\int_S \nabla \times \overline{A} \cdot d\overline{s} = \oint_L \overline{A} \cdot d\overline{L}$$

2. State coulombs law.

Ans. Coulombs law states that the force between any two point charges is directly proportional to the product of their magnitudes and inversely proportional to the square of the distance between them. It is directed along the line joining the two charges.

$$\overline{F}_{12} = \frac{Q_1 Q_2}{4\pi \in_0 R^2} \overline{a}_{R12}$$

3. State Gauss law for eelectric fields.

Ans. The total electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

4. Define electric flux.

Ans. The lines of electric force is electric flux.

5. Define electric flux density.

Ans. Electric flux density is defined as electric flux per unit area.

6. Define electric field intensity.

Ans. Electric field intensity is defined as the electric force per unit positive charge.

7. Name few applications of Gauss law in electrostatics.

Ans. Gauss law is applied to find the electric field intensity from a closed surface, i.e., Electric field can be determined for shell, two concentric shell or cylinders etc.

8. What is a point charge?

Ans. Point charge is one whose maximum dimension is very small in comparison with any other length.

9. Define linear charge density.

Ans. It is the charge per unit length.

10. Write poisson's and laplace's equations.

Ans. Poisson's eqn:

$$\nabla^2 V = \frac{-\rho_v}{\in}$$

Laplace's eqn:

$$\nabla^2 V = 0$$

11. Define potential difference.

Ans. Potential difference is defined as the work done in moving a unit positive charge from one point to another point in an electric field.

12. Define potential.

Ans. Potential at any point is defined as the work done in moving a unit positive charge from infinity to that point in an electric field.

13. Give the relation between electric field intensity and electric flux density.

Ans. $$\overline{D} = \in \overline{E} \text{ C/m}^2$$

14. Give the relationship between potential gradiant and electric field.

Ans. $$\overline{E} = -\nabla V$$

15. What is the physical significance of div D ?

Ans. $$\nabla . \overline{D} = -\rho_v$$

The divergence of a vector flux density is electric flux per unit volume leaving a small volume. This is equal to the volume charge density.

16. Define current density

Ans. Current density is defined as the current per unit area.

$$J = \frac{I}{A} \text{ Amp/m}^2$$

17. Write the point form of continuity equation and explain its significance.

Ans. $\therefore$ $$\nabla \cdot \overline{J} = -\frac{\partial \rho_v}{\partial t}$$

which is the continuity current equation and it's significance is:

The left side of the equation is the divergence of the Electric Current Density $(\bar{J})$. This is a measure of whether current is flowing into a volume (i.e., the divergence of $\bar{J}$ is positive if more current leaves the volume than enters).

Recall that current is the flow of electric charge. So if the divergence of $\bar{J}$ is positive, then more charge is exiting than entering the specified volume. If charge is exiting, then the amount of charge within the volume must be decreasing. This is exactly what the right side is a measure of - how much electric charge is accumulating or leaving in a volume. Hence, the continuity equation is about continuity - if there is a net electric current is flowing out of a region, then the charge in that region must be decreasing. If there is more electric current flowing into a given volume than exiting, then the amount of electric charge must be increasing.

18. Write the expression for energy density in electrostatic field.

Ans. $$w_E = \frac{1}{2} \in E^2$$

19. Write down the expression for capacitance between two parallel plates.

Ans. $$C = \frac{\in A}{d}$$

20. What is meant by displacement current?

Ans. Displacement current is the current flowing through the capacitor.

Multiple Choice Questions

1. Q_1 and Q_2 are two point charges, which are at a distance 8 cm apart. The force acting on Q_2 is given by $\bar{F}_{21} = \bar{a}_y 9 \times 10^{-12}$ N. Now we replace Q_2 with a charge of the same magnitude but opposite polarity, $Q_3 = -Q_2$, and we place Q_3 at a distance 24 cm away from Q_1. What is the vector F_{31} of the force acting on Q_3?

   (a) $\bar{F}_{31} = 3 \times 10^{-12} \bar{a}_y$ N (b) $\bar{F}_{31} = -3 \times 10^{-12} \bar{a}_y$ N

   (c) $\bar{F}_{31} = -1 \times 10^{-12} \bar{a}_y$ N (d) $\bar{F}_{31} = 1 \times 10^{-12} \bar{a}_y$ N

2. The intensity of the field due to a point charge Q_1 at a distance R_1 = 1 cm away from it is E_1 = 1 V/m. What is the intensity E_2 of the field of a charge $Q_2 = 4Q_1$ at a distance R_2 = 2 cm from it?

   (a) E_2 = 1 V/m (b) E_2 = 4 V/m

   (c) E_2 = 2 V/m (d) E_2 = ½ V/m

3. The intensity of the field due to a line charge p_{L1} at a distance r_1 = 1 cm away from it is E_1 = 1 V/m. What is the intensity E_2 of the field of the line charge p_{L2} = 4 at a distance r_2 = 2 cm from it?

   (a) E_2 = 1 V/m (b) E_2 = 4 V/m

   (c) E_2 = 2 V/m (d) E_2 = ½ V/m

4. Charge Q is uniformly distributed in a sphere of radius a_1. How is the charge density going to change if this same charge is now occupying a sphere of radius $a_2 = a_1/4$?

   (a) It will increase 4 times (b) It will increase 64 times

   (c) It will increase 16 times (d) It will increase 2 times

5. A line charge $p_L = 5 \times 10^{-3}$ C/m is located at $(x, y) = (0, 0)$, and is along the z-axis. Calculate the surface charge density p_s ($p_s > 0$) and the location x_p ($x_p > 0$) of an infinite planar charge distributed on the plane at $x = x_p$, so that the total field at the point P (0. 5 × 10^{-3}, 0) m, is zero.

   (a) $\rho_s = 1/(2\pi)$ C/m^2, $x_p = 5\times10^{-3}$ m (b) $\rho_s = 1/(2\pi)$ C/m^2, $\forall x_p$

   (c) $\rho_s = 1/\pi$ C/m^2, $x_p = 10\times10^{-3}$ m (d) $\rho_s = 1/\pi$ C/m^2, $\forall x_p$

6. The volume charge density associated with the electric displacement vector in spherical coordinates $\left(\sin\theta\sin\phi\, a_r + \cos\theta\sin\phi\, a_\phi + \cos\ \phi\, a_\phi\right)$ is

   (a) 0 (b) 1

   (c) Not compatible (d) sinθ

7. The divergence theorem
   (a) Relates a line integral to a surface integral
   (b) Holds for specific vector fields only
   (c) Works only for open surfaces
   (d) Relates a surface integral to a volume integral

8. The flux of a vector quantity crossing a closed surface
   (a) is always zero
   (b) is related to the quantity's component normal to the surface
   (c) is related to the quantity's component tangential to the surface
   (d) is not related in any way to the divergence of that vector quantity

9. The flux produced by a given set of fixed charges enclosed in a given closed region is
   (a) Dependent on the surface shape of the region, but not the volume
   (b) Dependent on the total volume of the region, but not the surface shape
   (c) Dependent on the ratio of volume to surface area of the region
   (d) Not dependent on any of these as long as the charges are inside the region

10. Consider charges placed inside a closed hemisphere. Consider the flux due to these charges through the curved regions (Flux A) and through the flat region (Flux B)
    (a) Flux A = Flux B
    (b) Flux A > Flux B
    (c) Flux A < Flux B
    (d) Not enough information to decide the relation between Flux A and Flux B

11. An electron ($q_e = 1.602 \times 10^{-19}$ C) leaves the cathode of a cathode ray tube (CRT) and travels in a uniform electrostatic field toward the anode, which is at a potential V_a = 500 V with respect to the cathode. What is the work W done by the electrostatic field involved in moving the electron from the cathode to the anode?
    (a) $W = 5\,\text{kJ}$ (b) $W = 8 \times 10^{-19}\,\text{J}$
    (c) $W = 8 \times 10^{-17}\,\text{J}$ (d) $W = 5\,\text{J}$

12. In the previous question, what is the electric field strength $E = |E|$ if the distance between the cathode and the anode is 10 cm?
    (a) $E = 5$ V/m (b) $E = 500$ V/m
    (c) $E = 50$ V/m (d) $E = 5$ kV/m

13. The electrostatic potential due to a point charge Q_1 at a distance r_1 = 1 cm away from it is V_1 = 1 V. What is the potential V_2 of a charge $Q_2 = 4Q_1$ at a distance r_2 = 2 cm from it?
    (a) $V_2 = 0.5$ V (b) $V_2 = 1$ V
    (c) $V_2 = 4$ V (d) $V_2 = 2$ V

14. The electrostatic potential due to a dipole $p_1 = p_1 a_2$ at a distance r_1 = 1 cm away from it along the z-axis, is V_1 = 1 V. What is the potential V_2 of a dipole $p_2 = 4p_1 a_Z$ at a distance r_2 = 2 cm from it along the z-axis?
    (a) $V_2 = 0.5$ V (b) $V_2 = 1$ V
    (c) $V_2 = 4$ V (d) $V_2 = 2$ V

15. The electrostatic potential $V = \dfrac{2\times10^{-3}}{\sqrt{\epsilon_0}}x$ V. Where x is measured in meters and ϵ_0 is the permittivity of vacuum, exists in a region of space (vacuum) in the shape of a parallelogram of size 10 × 10 × 1 cm. What is the electrostatic energy W_E stored in this region?

(a) $W_E = 2\times10^{-10}$ J (b) $W_E = 1\times10^{-9}$ J

(c) $W_E = 4\times10^{-10}$ J (d) $W_E = 3\times10^{-9}$ J

16. Which statement is not true?

(a) The static electric field in a conductor is zero

(b) The conductor surface is equipotential

(c) Zero tangential electric field on the surface of a conductor leads to zero potential difference between points on the surface

(d) The normally directed electrical field on the surface of a conductor is zero

17. The "skin" effect results in

(a) Current flowing in the entire volume as frequency increases

(b) Current flowing only near the surface as frequency increases

(c) Current flowing only near the surface as frequency decreases

(d) Current flowing near the surface at any frequency

18. As frequency increases, skin effect results in

(a) Decreased resistance

(b) Increased resistance

(c) No change in resistance

(d) Increase or decrease depending on material properties.

19. In a parallel-plate capacitor, the charge on the plates is C. What is the electric flux density magnitude D, if the area of each plate is $A = 10^{-4}$ m^2. Assume uniform field distribution.

(a) $D = 10^{-5}$ C/m^2 (b) $D = 10^{-5}/\epsilon_0$ C/m^2

(c) $D = 10^{-5}\,\epsilon_0$ C/m^2 (d) $D = 10^{-13}$ C/m^2

20. For the capacitor in Previous question, find the voltage between its plates, provided its capacitance is $C = 10$ pF.

(a) $V \simeq 885$ V (b) $V = 0$ V

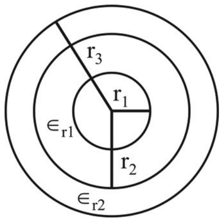
(c) $V = 100$ V (d) $V = 10^{-5}$ V

21. The capacitor in above Q no. 19 and 20 uses dielectric of permittivity $\in = \in_0$. The maximum allowable field intensity (*dielectric strength*) of this dielectric is E_{ds} = 3 MV/m. (If $E > E_{ds}$, the material breaks down.) What is the maximum voltage V_{max}, up to which the capacitor can operate safely (*its breakdown voltage*)?

    (a) $V_{max} = 885\,\text{V}$ (b) $V_{max} = 1000\,\text{V}$

    (c) $V_{max} = 3\times10^6\,\text{V}$ (d) $V_{max} = 265\,\text{V}$

22. A coaxial capacitor whose cross-section is shown in the figure below has a central conductor of radius r_1 and an outer conductor of radius r_3. The region between the two conductors consists of two regions: (i) the region $r_1 < p < r_2$ has a relative permittivity of $\varepsilon_{r1} = 2$ and (ii) the region $r_2 < p < r_3$ has a relative permittivity of $\varepsilon_{r2} = 1$. The radius r_2 is such that $r_2/r_1 = e^2$ and $r_3/r_2 = e$ where $e \approx 2.71$.



    What is the capacitance per unit length?

    (a) $C_1 = 4\pi \in_0$ (b) $C_1 = \pi \in_0$

    (c) $C_1 = 2\pi \in_0$ (d) $C_1 = \pi \in_0/2$

23. Poisson's and Laplace's equations are different in terms of

    (a) Definition of potential (b) Presence of non-zero charge

    (c) Boundary conditions on potential (d) No difference

Answers

1.	(c)	9.	(d)	17.	(b)
2.	(a)	10.	(a)	18.	(b)
3.	(c)	11.	(c)	19.	(a)
4.	(b)	12.	(d)	20.	(c)
5.	(d)	13.	(d)	21.	(d)
6.	(a)	14.	(b)	22.	(b)
7.	(d)	15.	(a)	23.	(b)
8.	(b)	16.	(d)		

Exercise Questions

1. State the Coulomb's law in SI units and indicate the parameters used in the equations with the aid of a diagram.
2. State Gauss's law. Using divergence theorem and Gauss's law, relate the density D to the volume charge density ρ_v.
3. Explain the following terms:
   (a) Homogeneous and isotropic medium and
   (b) Line, surface and volume charge distributions.
4. State and Prove Gauss's law. List the limitations of Gauss's law.
5. Express Gauss's law in both integral and differential forms. Discuss the salient features of Gauss's law.
6. Derive Poisson's and Laplace's equations starting from Gauss's law.
7. Using Gauss's law derive expressions for electric field intensity and electric flux density due to an infinite sheet of conductor of charge density ρ C/m.
8. Find the force on a charge of –100 mC located at P(2, 0, 5) in free space due to another charge 300 μC located at Q(1, 2, 3).
9. Find the force on a 100 μC charge at(0, 0, 3) m, if four like charges of 20 μC are located on X and Y axes at ± 4 m.
10. Derive an expression for the electric field intensity due to a finite length line charge along the Z-axis at an arbitrary point Q(x, y, z).
11. A point charge of 15 nC is situated at the origin and another point charge of –12 nC is located at the point (3, 3, 3) m. Find $\overline{E}$ and V at the point(0, –3, –3).
12. Obtain the expressions for the field and the potential due to a small Electric dipole oriented along Z-axis.
13. Define conductivity of a material. Explain the equation of continuity for time varying fields.
14. As an example of the solution of Laplace's equation, derive an expression for capacitance of a parallel plate capacitors.
15. In a certain region $\overline{J} = 3r^2 \cos\theta \overline{a}_r - r^2 \sin\theta \overline{a}_\theta \text{ A/m}$, find the current crossing the surface defined by $\theta = 30^o, 0 < \phi < 2\pi, 0 < r < 2\text{ m}$.