CHAPTER 1

INTRODUCTION

1.1 DESIGN AND ANALYSIS OF A COMPONENT

Mechanical design is the design of a component for optimum size, shape, etc., ***against failure*** under the application of operational loads. A good design should also minimise the cost of material and cost of production. Failures that are commonly associated with mechanical components are broadly classified as:

(a) Failure by breaking of brittle materials and fatigue failure (when subjected to repetitive loads) of ductile materials.

(b) Failure by yielding of ductile materials, subjected to non-repetitive loads.

(c) Failure by elastic deformation.

The last two modes cause change of shape or size of the component rendering it useless and, therefore, refer to functional or ***operational failure***. Most of the design problems refer to one of these two types of failures. Designing, thus, involves estimation of stresses and deformations of the components at different critical points of a component for the specified loads and boundary conditions, so as to satisfy operational constraints.

Design is associated with the calculation of dimensions of a component to withstand the applied loads and perform the desired function. ***Analysis*** is associated with the estimation of displacements or stresses in a component of assumed dimensions so that adequacy of assumed dimensions is validated. ***Optimum design*** is obtained by many iterations of modifying dimensions of the component based on the calculated values of displacements and/or stresses *vis-à-vis* permitted values and re-analysis.

An analytic method is applied to a model problem rather than to an ***actual physical problem***. Even many laboratory experiments use models. A ***geometric***

model for analysis can be devised after the physical nature of the problem has been understood. A model excludes superfluous details such as bolts, nuts, rivets, but includes all essential features, so that analysis of the model is not unnecessarily complicated and yet provides results that describe the actual problem with sufficient accuracy.

A geometric model becomes a ***mathematical model*** when its behaviour is described or approximated by incorporating restrictions such as homogeneity, isotropy, constancy of material properties and mathematical simplifications applicable for small magnitudes of strains and rotations.

For example, a plane truss (shown in Fig 1.1(a)) with different members joined by plates and multiple rivets is simplified by excluding less important joint details, assuming hinged conditions and homogeneous, isotropic material properties (as shown in Fig 1.1 (b)). In addition, self weight of members, which produce bending, is neglected in comparison with the large external loads. In some trusses, members overlapping on each other at the joints may be assumed co-planar.

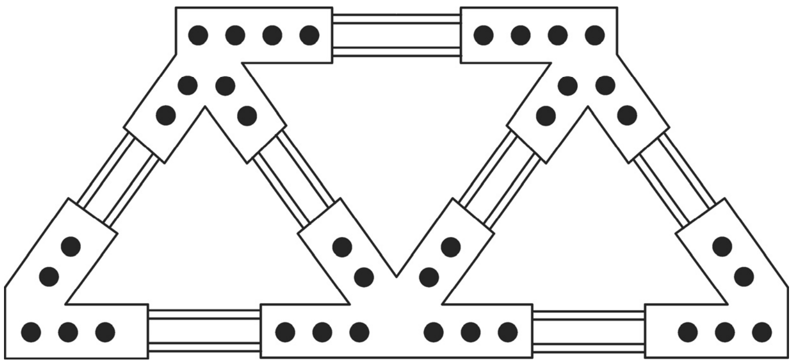


FIGURE 1.1(a) Physical structure of a plane truss

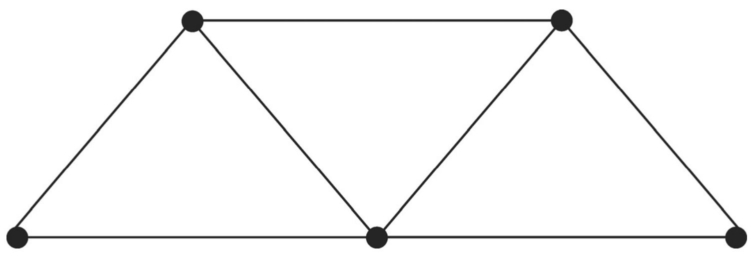


FIGURE 1.1(b) Geometric model of a plane truss

Several methods, such as method of joints for trusses, simple theory of bending, simple theory of torsion, analyses of cylinders and spheres for axi-symmetric pressure load etc., are available for designing/analysing simple components of a structure. These methods try to obtain exact solutions of second order partial differential equations and are based on several assumptions on sizes of the components, loads, end conditions, material properties, likely deformation pattern etc. Also, these methods are not amenable for generalisation and effective utilisation of the computer for repetitive jobs.

Strength of materials approach deals with a single beam member for different loads and end conditions (free, simply supported and fixed). In a space frame involving many such beam members, each member is analysed independently by an assumed distribution of loads and end conditions.

For example, in a 3-member structure (portal frame) shown in Fig. 1.2, the (horizontal) beam is analysed for deflection and bending stress by strength of materials approach considering its both ends simply supported. The load and moment reactions obtained at the ends are then used to calculate the deflections and stresses in the two columns separately.

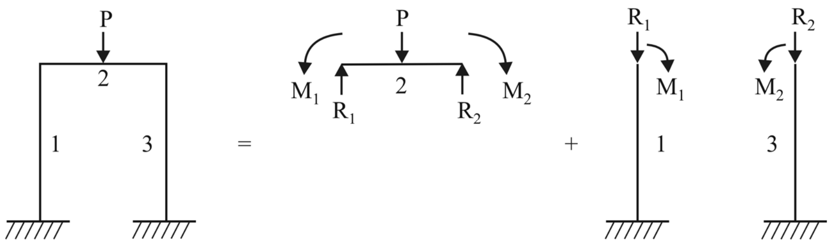


FIGURE 1.2 Analysis of a simple frame by strength of materials approach

Simple supports for the beam imply that the columns do not influence slope of the beam at its free ends (valid when bending stiffness of columns = 0 or the column is highly flexible). Fixed supports for the beam imply that the slope of the beam at its ends is zero (valid when bending stiffness of columns = ∞ or the column is extremely rigid). But, the ends of the horizontal beam are neither simply supported nor fixed. The degree of fixity or influence of columns on the slope of the beam at its free ends is based on a finite, non-zero stiffness value. Thus, the maximum deflection of the beam depends upon the relative stiffness of the beam and the columns at the two ends of the beam.

For example, in a beam of length ‘L’, modulus of elasticity ‘E’, moment of inertia ‘I’ subjected to a uniformly distributed load of ‘p’ (Refer Fig. 1.3).

Deflection, $\delta = \dfrac{5\,pL^4}{384\,EI}$ with simple supports at its two ends (case (a))

$= \dfrac{pL^4}{384\,EI}$ with fixed supports at its two ends (case (b))

Case (a) : Simple supports   Case (b) : Fixed supports

FIGURE 1.3 Deflection of a beam with different end conditions

If, in a particular case,
$L = 6$ m, $E = 2 \times 10^{11}$ N/m^2, Moment of inertia for beam $I_B = 0.48 \times 10^{-4}$ m^4
Moment of inertia for columns $I_C = 0.48 \times 10^{-4}$ m^4 and distributed load $p = 2$ kN/m,

$\delta_{max} = 3.515$ mm with simple supports at its two ends
and $\delta_{max} = 0.703$ mm with fixed supports at its two ends

whereas, deflection of the same beam, when analysed along with columns by FEM,

$\delta_{max} = 1.8520$ mm, when $I_B = I_C$ (Moments of inertia for beam & columns)
$= 1.0584$ mm, when $5\,I_B = I_C$
and $= 2.8906$ mm, when $I_B = 5\,I_C$

All the three deflection values clearly indicate presence of columns with finite and non-zero stiffness and, hence, the deflection values are in between those of beam with free ends and beam with fixed ends.

Thus, designing a single beam member of a frame leads to under-designing if fixed end conditions are assumed while it leads to over-designing if simple supports are assumed at its ends. Simply supported end conditions are, therefore, normally used for a conservative design in the conventional approach. Use of strength of materials approach for designing a component is, therefore, associated with higher factor of safety. The individual member method was acceptable for civil structures, where weight of the designed component is not a serious constraint. A more accurate analysis of discrete structures with few members is carried out by the potential energy approach. *Optimum beam design is achieved by analysing the entire structure which naturally considers finite stiffness of the columns, based on their dimensions and material, at its ends.* This approach is followed in the Finite Element Method (FEM).

1.2 APPROXIMATE METHOD VS. EXACT METHOD

An analytical solution is a mathematical expression that gives the values of the desired unknown quantity at any location of a body and hence is valid for an infinite number of points in the component. However, it is not possible to obtain analytical mathematical solutions for many engineering problems.

For problems involving complex material properties and boundary conditions, numerical methods provide approximate but acceptable solutions (with reasonable accuracy) for the unknown quantities – only at discrete or finite number of points in the component. Approximation is carried out in two stages :

(a) In the formulation of the mathematical model, w.r.t. the physical behaviour of the component. Example : Approximation of joint with multiple rivets at the junction of any two members of a truss as a pin joint, assumption that the joint between a column and a beam behaves like a simple support for the beam,.... The results are reasonably accurate far away from the joint. (Refer Fig. 1.1(a))

(b) In obtaining numerical solution to the simplified mathematical model. The methods usually involve approximation of a functional (such as Potential energy) in terms of unknown functions (such as displacements) at finite number of points. There are three broad categories:

   (i) **Weighted residual methods** such as Galerkin method, Collocation method, Least squares method, etc.

   (ii) **Variational method** (Rayleigh-Ritz method, FEM). FEM is an improvement of Rayleigh–Ritz method by choosing a variational function valid over a small element and ***not*** on the entire component, which will be discussed in detail later. These methods also use the principle of minimum potential energy.

   (iii) ***Principle of minimum potential energy :*** Among all possible kinematically admissible displacement fields (satisfying compatibility and boundary conditions) of a conservative system, the one corresponding to stable equilibrium state has minimum potential energy. For a component in static equilibrium, this principle helps in the evaluation of unknown displacements of deformable solids (continuum structures).

Some of these methods are explained here briefly to understand the historical development of analysis techniques.

1.3 WEIGHTED RESIDUAL METHODS

Most structural problems end up with differential equations. Closed form solutions are not feasible in many of these problems. Different approaches are suggested to obtain approximate solutions. One such category is the weighted residual technique. Here, an approximate solution, in the form $y = \Sigma N_i . C_i$ for i = 1 to n where C_i are the unknown coefficients or weights (constants) and N_i are functions of the independent variable satisfying the given kinematic boundary conditions, is used in the differential equation. Difference between the two sides of the equation with known terms, on one side (usually functions of the applied loads), and unknown terms, on the other side (functions of constants C_i), is called the residual, R. This residual value may vary from point to point in the component, depending on the particular approximate solution. Different methods are proposed based on how the residual is used in obtaining the best (approximate) solution. Three such popular methods are presented here.

(a) Galerkin Method

It is one of the weighted residual techniques. In this method, solution is obtained by equating the integral of the product of function N_i and residual R over the entire component to zero, for each N_i. Thus, the 'n' constants in the approximate solution are evaluated from the 'n' conditions $\int N_i . R . dx = 0$ for i = 1 to n. The resulting solution may match with the exact solution at some points of the component and may differ at other points. The number of terms N_i used for approximating the solution is arbitrary and depends on the accuracy desired. This method is illustrated through the following examples of beams in bending.

Example 1.1

Calculate the maximum deflection in a simply supported beam, subjected to concentrated load 'P' at the center of the beam. (Refer Fig. 1.4)

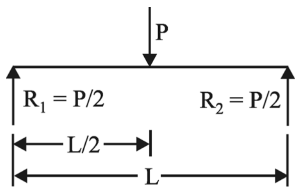


FIGURE 1.4

Solution

y = 0 at x = 0 and x = L are the kinematic boundary conditions of the beam. So, the functions N_i are chosen from $(x - a)^p.(x - b)^q$, with different positive integer values for p and q; and a = 0 and b = L.

(i) ***Model-1 (1-term approximation) :*** The deflection is assumed as

$$y(x) = N.c$$

with the function, $N = x(x - L)$,

which satisfies the end conditions y = 0 at x = 0 and y = 0 at x = L.

The load-deflection relation for the beam is given by

$$EI\left(\frac{d^2y}{dx^2}\right) = M$$

where $M = (P/2).x$ for $0 \leq x \leq L/2$

and $M = (P/2).x - P.[x - (L/2)] = (P/2).(L - x)$ for $L/2 \leq x \leq L$

Thus, taking $y = x.(x - L).c$,

$$\frac{d^2y}{dx^2} = 2c$$

and the residual of the equation, $R = EI\left(\frac{d^2y}{dx^2}\right) - M = EI\ .\ 2c - M$

Then, the unknown constant 'c' in the function 'N' is obtained from

$$\int_0^{\frac{L}{2}} N.R.dx + \int_{\frac{L}{2}}^{L} N.R.dx = 0$$

(two integrals needed, since expression for M changes at x = $\frac{L}{2}$)

$$\int_0^{\frac{L}{2}} [x.(x-L)].\left[EI.2c - \left(\frac{P}{2}\right).x\right]dx + \int_{\frac{L}{2}}^{L} [x.(x-L)].\left[EI.2c - \left(\frac{P}{2}\right).(L-x)\right]dx = 0$$

$$\Rightarrow c = \frac{5\ PL}{64\ EI}$$

Therefore, $y = x(x - L).\ \frac{5\ PL}{64\ EI}$

At $\quad x = \dfrac{L}{2}, \qquad y = y_{max} = -\dfrac{5\,PL^3}{256\,EI}$ or $\dfrac{-PL^3}{51.2\,EI}$

This approximate solution is close to the exact solution of $\dfrac{-PL^3}{48\,EI}$ obtained by double integration of EI $\left(\dfrac{d^2y}{dx^2}\right) = M = \left(\dfrac{P}{2}\right).x$, with appropriate end conditions.

(ii) ***Model-2 (2-term approximation):*** The deflection is assumed as

$$y(x) = N_1.c_1 + N_2.c_2$$

with the functions $N_1 = x(x - L)$ and $N_2 = x.(x - L)^2$

which satisfy the given end conditions.

Thus, taking $\quad y = x.(x - L).c_1 + x.(x - L)^2.c_2$,

$$\left(\frac{d^2y}{dx^2}\right) = 2c_1 + 2.(3x - 2L).c_2$$

and the residual of the equation,

$$R = EI.\left(\frac{d^2y}{dx^2}\right) - M = EI.[2c_1 + 2.(3x - 2L).c_2] - M$$

where $\quad M = (P/2).x \qquad$ for $\quad 0 \le x \le L/2$

and $\quad M = (P/2).x - P.[x - (L/2)] = (P/2).(L - x) \qquad$ for $\quad L/2 \le x \le L$

Then, the unknown constants 'c_1' and 'c_2' in the functions 'N_i' are obtained from

$$\int_0^L N_1.R.dx = \int_0^{\frac{L}{2}} [x.(x - L)]\left\{EI.[2c_1 + 2.(3x - 2L).c_2] - \left(\frac{P}{2}\right).x\right\} dx$$

$$+ \int_{\frac{L}{2}}^{L} [x.(x - L)]\left\{EI.[2c_1 + 2.(3x - 2L).c_2] - \left(\frac{P}{2}\right).(L - x)\right\} dx = 0$$

and $$\int_0^L N_2.R.dx = \int_0^{\frac{L}{2}} \left[x.(x-L)^2\right].\left\{EI.[2c_1 + 2.(3x-2L).c_2] - \left(\frac{P}{2}\right).x\right\}dx$$

$$+ \int_{\frac{L}{2}}^{L} \left[x.(x-L)^2\right]\left\{EI.[2c_1 + 2.(3x-2L).c_2] - \left(\frac{P}{2}\right).(L-x)\right\}dx = 0$$

Simplifying these equations, we get

$$2c_1 - c_2.L = \frac{5PL}{16EI} \qquad \text{and} \qquad 5c_1 - 4c_2.L = \frac{75PL}{192EI}$$

Solving these two simultaneous equations, we get

$$c_1 = \frac{55PL}{192EI} \qquad \text{and} \qquad c_2 = \frac{25P}{96EI}$$

Thus, we get

$$y = x(x-L).\ \frac{55PL}{192EI} + x(x-L)^2.\ \frac{25P}{96EI}$$

and at $x = L/2$, $y = y_{max} = \dfrac{PL^3(-55+25)}{4 \times 192EI}$ or $-\dfrac{PL^3}{25.6EI}$

Note : The bending moment M is a function of x. The exact solution of y should be a minimum of 3rd order function so that $\dfrac{d^2y}{dx^2} = \dfrac{M}{EI}$ is a function of x.

(b) Collocation Method

In this method, also called as the ***point collocation method***, the residual is equated to zero at 'n' select points of the component other than those at which the displacement value is specified, where 'n' is the number of unknown coefficients in the assumed displacement field, i.e., $R(\{c\},x_i) = 0$ for i = 1, ..n. It is also possible to apply collocation method on some select surfaces or volumes. In that case, the method is called ***sub-domain collocation method.***

i.e., $\int R(\{c\},x_j).dS_j = 0$ for j = 1, ..n

or $\int R(\{c\},x_k).\ dV_k = 0$ for k = 1, ..n

These methods also result in 'n' algebraic simultaneous equation in 'n' unknown coefficients, which can be easily evaluated.

The simpler of the two for manual calculation, point collocation method, is explained better through the following example.

Example 1.2

Calculate the maximum deflection in a simply supported beam, subjected to concentrated load 'P' at the center of the beam. (Refer Fig. 1.5)

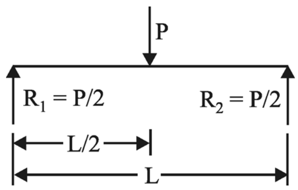


FIGURE 1.5

Solution

y = 0 at x = 0 and x = L are the kinematic boundary conditions of the beam. So, the functions N_i are chosen from $(x - a)^p.(x - b)^q$, with different positive integer values for p and q; and a = 0 and b = L.

(i) ***Model-1 (1-term approximation):*** The deflection is assumed as

$$y(x) = N.c$$

with the function $N = x(x - L)$,

which satisfies the end conditions y = 0 at x = 0 and y = 0 at x = L

The load-deflection relation for the beam is given by

$$EI\left(\frac{d^2y}{dx^2}\right) = M$$

where $M = (P/2).x$ for $0 \le x \le L/2$

and $M = (P/2).x - P.[x - (L/2)] = (P/2).(L - x)$ for $L/2 \le x \le L$

Thus, taking $y = x.(x - L).c$, $\dfrac{d^2y}{dx^2} = 2c$

and the residual of the equation, $R = EI\left(\dfrac{d^2y}{dx^2}\right) - M = EI\left(\dfrac{d^2y}{dx^2}\right) - \left(\dfrac{P}{2}\right).x$

Then, the unknown constant 'c' in the function 'N' is obtained by choosing the value of residual at some point, say x = L/2, as zero.

i.e., $$R(c, x) = EI.2c - \left(\frac{P}{2}\right).x = 0 \text{ at } x = \frac{L}{2} \Rightarrow c = \frac{PL}{8EI}$$

Therefore, $y = x(x - L).\dfrac{PL}{8EI}$

At $x = \dfrac{L}{2}$, $y = y_{max} = -\dfrac{PL^3}{32EI}$

(ii) ***Model-2 (2-term approximation) :*** The deflection is assumed as

$$y(x) = N_1.c_1 + N_2.c_2$$

with the functions $N_1 = x(x - L)$ and $N_2 = x.(x - L)^2$,
which satisfy the given end conditions.
Thus, taking $y = x.(x - L).c_1 + x.(x - L)^2.c_2$,

$$\frac{d^2y}{dx^2} = 2c_1 + 2.(3x - 2L).c_2$$

and the residual of the equation,

$$R = EI.\left(\frac{d^2y}{dx^2}\right) - M = EI.[2c_1 + 2.(3x - 2L).c_2] - M$$

where $M = (P/2).x$ for $0 \le x \le L/2$
and $M = (P/2).x - P.[x - (L/2)] = (P/2).(L - x)$ for $L/2 \le x \le L$

Then, the unknown constants 'c_1' and 'c_2' in the functions 'N_i' are obtained from

$$R(\{c\},x) = EI.[2c_1 + 2.(3x - 2L).c_2] - (P/2).x = 0 \quad \text{at } x = L/4$$

and $R(\{c\},x) = EI.[2c_1 + 2.(3x - 2L).c_2] - (P/2).(L - x)\} = 0$ at $x = 3L/4$

or $4c_1 - 5L.c_2 = \dfrac{PL}{4EI}$ and $4c_1 + L.c_2 = \dfrac{PL}{4EI}$

$$\Rightarrow c_1 = \frac{PL}{16EI} \text{ and } c_2 = 0$$

$$\Rightarrow y_{max} = -\frac{PL^3}{64EI} \text{ at } x = \frac{L}{2}$$

Choosing some other collocation points, say $x = \dfrac{L}{3}$ and $x = \dfrac{2L}{3}$,

$$c_1 - L.c_2 = \frac{PL}{12EI} \text{ and } c_1 + 0 = \frac{PL}{12EI}$$

$$\Rightarrow c_1 = \frac{PL}{12EI} \text{ and } c_2 = 0$$

At $x = \frac{L}{2}$, $y_{max} = \frac{-PL^3}{48EI}$, which matches exactly with closed form solution

(c) Least Squares Method

In this method, integral of the residual over the entire component is minimized. i.e., $\frac{\partial I}{\partial c_i} = 0$ for $i = 1, ..n,$

$$\text{where } I = \int [R(\{a\}, x)]^2 .dx$$

This method also results in 'n' algebraic simultaneous equations in 'n' unknown coefficients, which can be easily evaluated.

Example 1.3

Calculate the maximum deflection in a simply supported beam, subjected to concentrated load 'P' at the center of the beam. (Refer Fig. 1.6)

Solution

Again, y = 0 at x = 0 and y = 0 at x = L are the kinematic boundary conditions of the beam. So, the functions N_i are chosen from $(x - a)^p.(x - b)^q$, with different positive integer values for p and q.

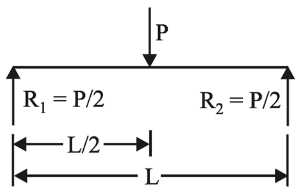


FIGURE 1.6

1-term approximation: The deflection is again assumed as $y(x) = N.C$,

with the function $N = x(x - L)$,

which satisfies the end conditions $y = 0$ at $x = 0$ and $y = 0$ at $x = L$

The load-deflection relation for the beam is given by

$$EI\left(\frac{d^2y}{dx^2}\right) = M$$

where $M = (P/2).x$ for $0 \le x \le L/2$

and $M = (P/2).x - P.[x - (L/2)] = (P/2).(L - x)$ for $L/2 \le x \le L$

Thus, taking $y = x.(x - L).c$, $\dfrac{d^2y}{dx^2} = 2c$

and the residual of the equation, $R = EI\left(\dfrac{d^2y}{dx^2}\right) - M = EI \,.\, 2c - M$

Then, $I = \int [R(\{c\},x)]^2.dx$ and the constant 'c' in the function y(x) is obtained from

$$\frac{\partial I}{\partial a_i} = \frac{\partial}{\partial c_i}\int_0^{\frac{L}{2}}\left[\left[R(\{c\},x)\right]^2 - \left(\frac{P}{2}\right).x\right]dx +$$

$$\frac{\partial}{\partial c_i}\int_{\frac{L}{2}}^{L}\left[\left[R(\{c\},x)\right]^2 - \left(\frac{P}{2}\right).(L-x)\right]dx = 0$$

$$\Rightarrow\ c = \frac{PL}{8EI} \quad \text{and} \quad y_{max} = \frac{-PL^3}{32EI} \quad \text{at} \quad x = \frac{L}{2}$$

1.4 VARIATIONAL METHOD OR RAYLEIGH - RITZ METHOD

This method involves choosing a displacement field over the entire component, usually in the form of a polynomial function, and evaluating unknown coefficients of the polynomial for minimum potential energy. It gives an approximate solution. Practical application of this method is explained here through three different examples, involving

(a) uniform bar with concentrated load,
(b) bar of varying cross section with concentrated load, and
(c) uniform bar with distributed load (self-weight).

Example 1.4

Calculate the displacement at node 2 of a fixed beam shown in Fig. 1.7, subjected to an axial load 'P' at node 2.

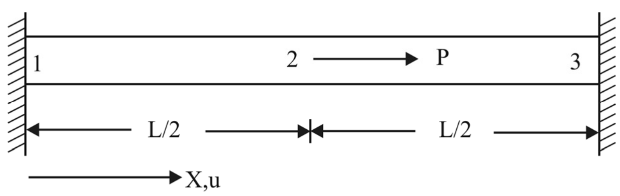


FIGURE 1.7

Solution

Method - 1

The total potential energy for the linear elastic one-dimensional rod with built-in ends, when body forces are neglected, is

$$\pi = \frac{1}{2}\int EA\left(\frac{du}{dx}\right)^2 dx - P\,u_2$$

Let us assume $u = a_1 + a_2x + a_3x^2$ as the polynomial function for the displacement field.

Kinematically admissible displacement field must satisfy the natural boundary conditions

$u = 0$ at $x = 0$ which implies $a_1 = 0$

and $u = 0$ at $x = L$ which implies $a_2 = -a_3 L$

At $x = \frac{L}{2}$, $u_2 = a_2\left(\frac{L}{2}\right) + a_3\left(\frac{L}{2}\right)^2 = -a_3\frac{L^2}{4}$

Therefore,

$$\pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - Pu_2$$

$$= \frac{1}{2}\int_0^L EA(a_2 + 2a_3x)^2 dx - P\left(-a_3\frac{L^2}{4}\right)$$

$$= \frac{1}{2}EAa_3^2\int_0^L (2x - L)^2 dx + P\,a_3\frac{L^2}{4}$$

$$= EAa_3^2\frac{L^3}{6} + P\,a_3\frac{L^2}{4}$$

For stable equilibrium, $\frac{\partial\pi}{\partial a_3} = 0$ gives $a_3 = \frac{-3P}{4EAL}$

Displacement at node 2, $u_2 = -\frac{a_3 L^2}{4} = \frac{3PL}{16AE}$

$$= \left(\frac{3}{4}\right).\left(\frac{PL}{4AE}\right)$$

It differs from the exact solution by a factor of $\frac{3}{4}$. Exact solution is obtained when a piece-wise polynomial interpolation is used in the assumption of displacement field, u.

Stress in the bar, $\sigma = E\left(\frac{du}{dx}\right) = E(a_2 + a_3 x) = E(x - L)a_3$

$$= \frac{-3\,PE(x-L)}{4\,EAL} = \frac{3\,P(L-x)}{4\,AL}$$

$$= +\left(\frac{3}{4}\right)\left(\frac{P}{A}\right) \text{ at } x = 0 \quad \text{and} \quad -\left(\frac{3}{4}\right)\left(\frac{P}{A}\right) \text{ at } x = L$$

or $$\pm\left(\frac{3}{2}\right)\left(\frac{P}{2A}\right)$$

Due to the assumption of a quadratic displacement field over the system, stress is found to vary along the length of the bar. However, stress is expected to be constant (tensile from 1 to 2 and compressive from 2 to 3). Hence, the solution is not exact.

Method - 2

In order to compare the accuracy of the solution obtained by Rayleigh-Ritz method, the beam is analysed considering it to be a system of two springs in series as shown in Fig. 1.8 and using the stiffness of the axially loaded bar in the potential energy function.

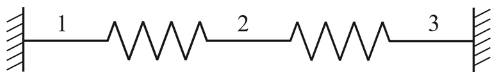


FIGURE 1.8

The stiffness of each spring is obtained from

$$k_1 = k_2 = K = \frac{P}{u} = \frac{(\sigma.A)}{\left[\varepsilon.\left(\frac{L}{2}\right)\right]} = \frac{2AE}{L}$$

Total potential energy of the system is given by

$$\pi = \left(\frac{1}{2}k_1 u_2^2 + \frac{1}{2}k_2 u_2^2\right) + (-Pu_2) = K.u_2^2 - Pu_2$$

For equilibrium of this 1-DOF system,

$$\frac{\partial \pi}{\partial u_2} = 2K.u_2 - P = 0$$

or $$u_2 = \frac{P}{2K} = \frac{PL}{4AE}$$

Stress in the beam is given by,

$$\sigma = E\,\varepsilon = E.\left[\frac{u_2}{\left(\frac{L}{2}\right)}\right] = \frac{2Eu_2}{L} = \frac{P}{2A}$$

The displacement at 2 by Rayleigh-Ritz method differs from the exact solution by a factor of $\frac{3}{4}$, while the maximum stress in the beam differs by a factor of $\frac{3}{2}$. The stresses obtained by this approximate method are thus on the conservative side. Exact solution is obtained when a piece-wise polynomial interpolation is used in the assumption of displacement field, u. The results are plotted in Fig. 1.9.

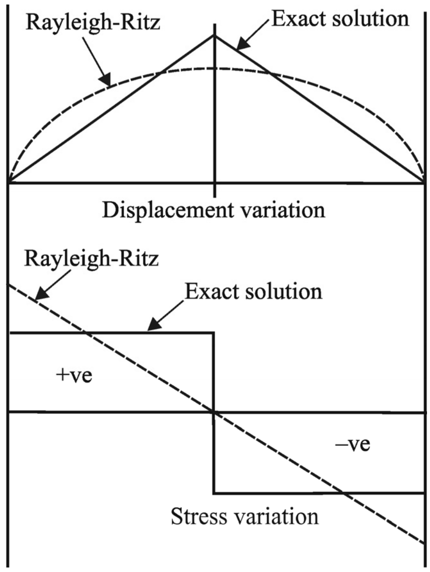


FIGURE 1.9

Method - 3

If the assumed displacement field is confined to a single element or segment of the component, it is possible to choose a more accurate and convenient polynomial. This is done in finite element method (FEM). Since total potential energy of each element is positive, minimum potential energy theory for the entire component implies minimum potential energy for each element. Stiffness matrix for each element is obtained by using this principle and these matrices for all the elements are assembled together and solved for the unknown displacements after applying boundary conditions. A more detailed presentation of FEM is provided in chapter 4.

Applying this procedure in the present example, let the displacement field in each element of the 2-element component be represented by $u = a_0 + a_1.x$. With this assumed displacement field, stiffness matrix of each axial loaded element of length (L/2) is obtained as

$$[K]=\left(\frac{2AE}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{and } \{P\} = [K]\ \{u\}$$

The assembled stiffness matrix for the component with two elements is then obtained by placing the coefficients of the stiffness matrix in the appropriate locations as

$$\left(\frac{2AE}{L}\right)\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

Applying boundary conditions $u_1 = 0$ and $u_3 = 0$, we get

$$\left(\frac{2AE}{L}\right)2\,u_2 = P_2 = P \ \Rightarrow\ u_2 = \frac{PL}{4AE} \text{ which is the exact solution.}$$

The potential energy approach and Rayleigh-Ritz method are now of only academic interest. FEM is a better generalisation of these methods and extends beyond discrete structures.

Examples of Rayleigh-Ritz method, with variable stress in the members

These examples are referred again in higher order 1-D truss elements, since they involve stress or strain varying along the length of the bar.

Example 1.5

Calculate displacement at node 2 of a tapered bar, shown in Fig 1.10, with area of cross-section A_1 at node 1 and A_2 at node 2 subjected to an axial tensile load 'P'.

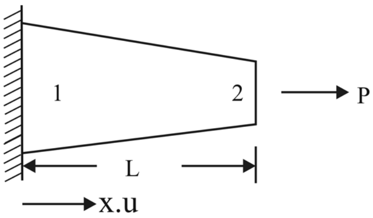


FIGURE 1.10

Solution

Different approximations are made for the displacement field and comparison made, in order to understand the significance of the most reasonable assumption.

(a) Since the bar is identified by 2 points, let us choose a first order polynomial (with 2 unknown coefficients) to represent the displacement field. Variation of A along the length of the bar adds additional computation. Let $A(x) = A_1 + (A_2 - A_1).x/L$

Let $u = a_1 + a_2.x$ At $x = 0,\ u = a_1 = 0$

Then, $u = a_2.x$; $\dfrac{du}{dx} = a_2$ and $u_2 = a_2.L$

Therefore,

$$\pi = \frac{1}{2}\int_0^L EA(x).\left(\frac{du}{dx}\right)^2 dx - P\,u_2$$

$$= \frac{1}{2}\int_0^L E\left[A_1 + (A_2 - A_1).\frac{x}{L}\right] a_2^2\, dx - P.a_2.L$$

$$= \frac{1}{2}E\left[A_1 L + (A_2 - A_1).\frac{L}{2}\right] a_2^2 - P.a_2.L$$

$$= \frac{1}{2}E\left[(A_1 + A_2).\frac{L}{2}\right] a_2^2 - P.a_2.L$$

For stable equilibrium, $\dfrac{\partial \pi}{\partial a_2} = E\left[(A_1 + A_2).\dfrac{L}{2}\right] a_2 - P.L = 0$

from which a_2 can be evaluated as, $a_2 = \dfrac{P}{\left[\dfrac{E(A_1 + A_2)}{2}\right]}$

Then, $$u_2 = a_2.L = \frac{P.L}{\left[\dfrac{E.(A_1 + A_2)}{2}\right]}$$

and $$\sigma_1 = \sigma_2 = E\left(\frac{du}{dx}\right) = E\,a_2 = \frac{P}{\left[\dfrac{(A_1 + A_2)}{2}\right]}$$

For the specific data of A_1 = 40 mm^2, A_2 = 20 mm^2 and L = 200 mm, we obtain,

$$u_2 = \frac{6.667\,P}{E} \quad \text{and} \quad \sigma_1 = \sigma_2 = 0.0333\,P$$

(b) Choosing displacement field by a first order polynomial gave constant strain (first derivative) and hence constant stress. Since a tapered bar is expected to have a variable stress, it is implied that the displacement field should be expressed by a polynomial of minimum 2^{nd} order. Therefore, the solution is repeated with

$$u = a_1 + a_2.x + a_3.x^2 \qquad \text{At} \quad x = 0, u = a_1 = 0$$

Then, $u = a_2.x + a_3.x^2$; $\dfrac{du}{dx} = a_2 + 2a_3.x$ and $u_2 = a_2.L + a_3.L^2$

Therefore, $$\pi = \frac{1}{2}\int_0^L EA(x).\left(\frac{du}{dx}\right)^2 dx - P\,u_2$$

$$= \frac{1}{2}\int_0^L E\left[A_1 + (A_2 - A_1).\frac{x}{L}\right](a_2 + 2a_3x)^2 dx - P\left(a_2.L + a_3.L^2\right)$$

For stable equilibrium,

$$\frac{\partial \pi}{\partial a_2} = 0 \Rightarrow 3\,a_2\,(3A_1 - A_2) + 2a_3\,L\,(5A_1 - 2A_2) = \frac{6P}{E}$$

and $$\frac{\partial \pi}{\partial a_3} = 0 \Rightarrow a_2\,(5A_1 - 2A_2) + a_3\,L\,(7A_1 - 3A_2) = \frac{3P}{E}$$

For the specific data of A_1 = 40 mm^2, A_2 = 20 mm^2 and L = 200 mm, we obtain,

$$u_2 = a_2.L + a_3.L^2 = 6.652\,\frac{P}{E}$$

$$\sigma_1 = a_2 E \qquad = 0.0339\ P$$

and $\quad \sigma_2 = E(a_2 + 2a_3 L) = 0.03518\ P$

(c) This problem can also be solved by assuming a 2nd order displacement function, satisfying a linearly varying stress along the length but with 2 unknown coefficients as

$$u = a_1 + a_2.x^2 \qquad \text{At } x = 0,\ u = a_1 = 0$$

Then, $\quad u = a_2.x^2;\ \dfrac{du}{dx} = 2a_2.x$ and $u_2 = a_2.L^2$

Therefore,

$$\pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - Pu_2$$

$$= \frac{1}{2}\int_0^L E\left[A_1 + (A_2 - A_1).\frac{x}{L}\right](2a_2\,x)^2\,dx - P\left(a_2.L^2\right)$$

$$= \frac{1}{2}E\left[\frac{\left[A_1 L^3\right]}{3} + \frac{(A_2 - A_1).L^3}{4}\right]\left(4a_2^2\right) - P\left(a_2.L^2\right)$$

$$= a_2^2\,L\,E\,\frac{(A_1 + 3A_2)}{6} - P\,a_2.L^2$$

For stable equilibrium, $\dfrac{\partial\pi}{\partial a_2} = \dfrac{a_2 L\,E\,(A_1 + 3A_2)}{3 - P\,L^2} = 0$ which gives an expression for a_2 as

$$a_2 = \frac{P\,L}{\left[\dfrac{E\,(A_1 + 3A_2)}{3}\right]}$$

For the same set of data for A_1, A_2 and L, we get

$$u_2 = a_2.L^2 = \frac{6\,P}{E}$$

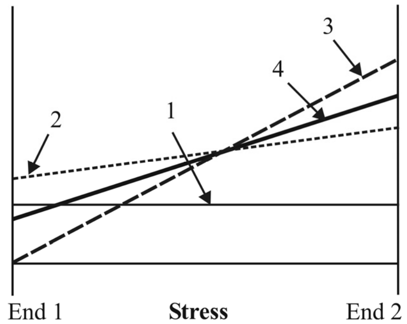
$$\sigma_1 = E.\in_1 = E.\left(\frac{du}{dx}\right)_{x=0}$$

$$= E.(2a_2 x)_{x=0} = 0$$

$$\sigma_2 = E.(2a_2 x)_{x=L} = 2Ea_2 L = 0.06\ P$$

Sr.No.	Displacement polynomial	Displacement, u_2	Stress at 1, σ_1	Stress at 2, σ_2
1	$u = a_1 + a_2.x$	6.667 P/E	0.0333 P	0.0333 P
2	$u = a_1 + a_2.x + a_3.x^2$	6.652 P/E	0.0339 P	0.03518 P
3	$u = a_1 + a_2.x^2$	6.0 P/E	0.0	0.06 P
4	$\sigma_i = P/A_i$ (Exact solution)		0.025 P	0.05 P

These three assumed displacement fields gave different approximate solutions. These are plotted graphically here, for a better understanding of the differences. Exact solution depends on how closely the assumed displacement field matches with the actual displacement field.



The most appropriate displacement field should necessarily include constant term, linear term and then other higher order terms.

Example 1.6

Calculate the displacement at node 2 of a vertical bar, shown in Fig. 1.11, due to its self-weight. Let the weight be w N/m of length.

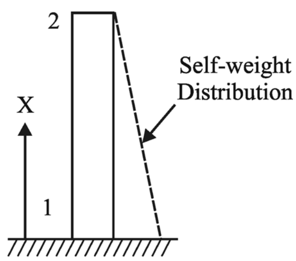


FIGURE 1.11

Solution

Since the load is distributed, varying linearly from zero at the free end to a maximum at the fixed end, it implies that the stress also varies linearly from the free end to fixed end. As shown in the last example, therefore, a quadratic displacement is the most appropriate. However, work potential needs to be calculated through integration of product of varying load and corresponding displacement, along the length.

(a) Let $u = a_1 + a_2.x + a_3.x^2$ At $x = 0$, $u_1 = a_1 = 0$

$$\frac{du}{dx} = a_2 + 2a_3.x$$

Since applied load is zero at the free end,

Strain at x = L, $\left(\frac{du}{dx}\right)_2 = a_2 + 2a_3.L = 0 \Rightarrow a_2 = -2a_3L$

Then, $u = a_3.(x^2 - 2Lx)$; and $\frac{du}{dx} = 2a_3.(x - L)$

Let $P = -w(L - x)$ acting along –ve x-direction

Therefore,

$$\pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L P\, du$$

$$= \frac{1}{2}\int_0^L EA.[2a_3.(x - L)]^2 dx - \int_0^L [-w(L - x)].2a_3.(x - L)\,dx$$

$$= \frac{2EA.a_3^2.L^3}{3} - \frac{2w.a_3.L^3}{3}$$

For stable equilibrium, $\frac{\partial\pi}{\partial a_3} = 0 \Rightarrow a_3 = \frac{w}{2EA}$

At x = L, $u_2 = -a_3.L^2 = \frac{-w.L^2}{2EA}$

Stress, $\sigma = E.\frac{du}{dx} = E.[2a_3.(x - L)] = \left(\frac{w}{A}\right).(x - L)$

At x = 0, $\sigma_1 = -\left(\frac{w.L}{A}\right)$ compressive

And at x = L, $\sigma_2 = 0$

Example 1.7

Calculate the displacement at node 2 of a vertical bar supported at both ends, shown in Fig. 1.12, due to its self-weight. Let the weight be w N/m of length.

Solution

As explained in the last example, a quadratic displacement is the most appropriate to represent linearly varying stress along the bar.

(a) Let $u = a_1 + a_2.x + a_3.x^2$ At $x = 0,\ u_1 = a_1 = 0$

At $x = L$, $u_2 = a_2.L + a_3.L^2 = 0$ $\Rightarrow a_2 = -a_3L$

Then, $u = a_3.(x^2 - Lx)$

and $\dfrac{du}{dx} = a_3.(2x - L)$

Let $P = -w(L - x)$ acting along –ve x-direction

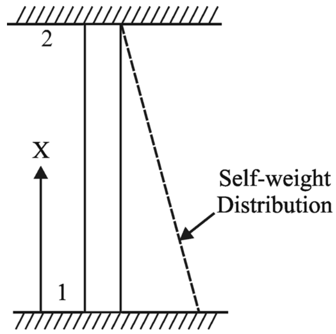


FIGURE 1.12

Therefore,

$$\pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L P\,du$$

$$= \frac{1}{2}\int_0^L EA.[a_3.(2x-L)]^2\,dx + \int_0^L w(L-x).a_3.(2x-L)dx$$

$$= \frac{1}{2}EA.a_3^2.\frac{L^3}{3} - 2w.a_3.\frac{L^3}{3}$$

For stable equilibrium,

$$\frac{\partial \pi}{\partial a_3} = 0 \;\Rightarrow\; a_3 = \frac{2w}{EA}$$

At $x = \frac{L}{2}$, $u = -a_3.\frac{L^2}{4} = -w.\frac{L^2}{2EA}$

Stress, $\sigma = E.\frac{du}{dx} = E.[a_3.(2x - L)] = \left(\frac{2w.}{A}\right).(2x - L)$

At $x = 0$, $\sigma_1 = -(2w.L/A)$ compressive

and at $x = L$, $\sigma_2 = (2w.L/A)$ tensile

1.5 PRINCIPLE OF MINIMUM POTENTIAL ENERGY

The total potential energy of an elastic body (π) is defined as the sum of total strain energy (U) and work potential (W).

i.e., $\pi = U + W$,

where $U = \left(\frac{1}{2}\right)\int_V \sigma\,\varepsilon\,dV$

and $W = -\int_V u^T F\,dV - \int_S u^T\,T\,dS - \sum u_i\,P_i$

Here, 'F' is the distributed body force, 'T' is the distributed surface traction force and 'P_i' are the concentrated loads applied at points i = 1, ..n. One or more of them may be acting on the component at any instant.

For a bar with axial load, if stress σ and strain ε are assumed uniform throughout the bar,

$$U = \left(\frac{1}{2}\right)\sigma\,\varepsilon\,v = \left(\frac{1}{2}\right)\sigma\,\varepsilon\,A\,L = \left(\frac{1}{2}\right)(\sigma\,A)\,(\varepsilon\,L) = \left(\frac{1}{2}\right)P\delta = \left(\frac{1}{2}\right)k\,\delta^2 \qquad \text{.....(1.1)}$$

The work potential, $W = -\int q^T\,F\,dV - \int q^T\,T\,ds - \sum u_i^T\,P_i$ **.....(1.2)**

for the body force, surface traction and concentrated loads, respectively.

Application of this method is demonstrated through the following simple examples.

Example 1.8

Calculate the nodal displacements in a system of four springs shown in Fig. 1.13.

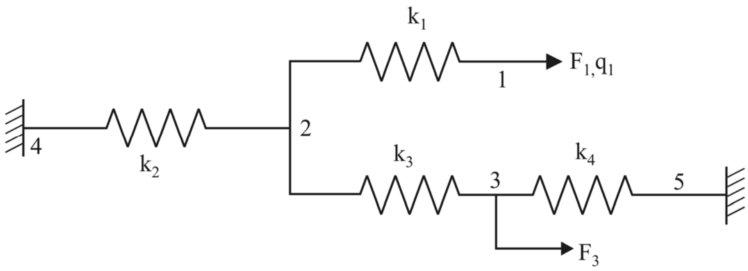


FIGURE 1.13 Example of a 5-noded spring system

Solution

The total potential energy is given by

$$\pi = \left(\frac{1}{2}k_1\delta_1^2 + \frac{1}{2}k_2\delta_2^2 + \frac{1}{2}k_3\delta_3^2 + \frac{1}{2}k_4\delta_4^2\right) + \left(-F_1q_1 - F_3q_3\right)$$

where, q_1, q_2, q_3 are the three unknown nodal displacements.

At the fixed points

$q_4 = q_5 = 0$

Extensions of the four springs are given by

$\delta_1 = q_1 - q_2$; $\delta_2 = q_2$

$\delta_3 = q_3 - q_2$; $\delta_4 = -q_3$

For equilibrium of this 3-DOF system,

$$\frac{\partial \pi}{\partial q_i} = 0 \text{ for i = 1, 2, 3}$$

or

$$\frac{\partial \pi}{\partial q_1} = k_1(q_1 - q_2) - F_1 = 0$$

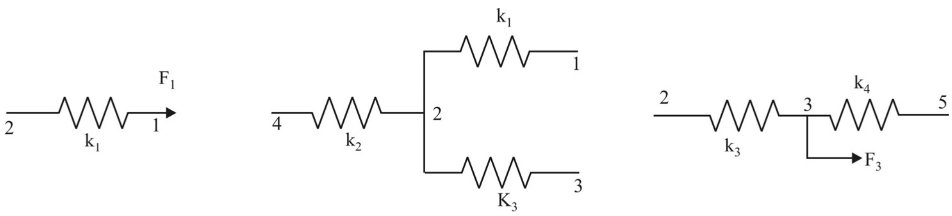
$$\frac{\partial \pi}{\partial q_2} = -k_1(q_1 - q_2) + k_2q_2 - k_3(q_3 - q_2) = 0$$

$$\frac{\partial \pi}{\partial q_3} = k_3(q_3 - q_2) + k_4q_3 - F_3 = 0$$

These three equilibrium equations can be rewritten and expressed in matrix form as

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ F_2 \end{Bmatrix}$$

Considering free body diagrams of each node separately, represented by the following figures,



the equilibrium equations are $k_1\delta_1 = F_1$

$$k_2\delta_2 - k_1\delta_1 - k_3\delta_3 = 0$$

$$k_3\delta_3 - k_4\delta_4 = F_3$$

These equations, expressed in terms of nodal displacements q, are similar to the equations obtained earlier by the potential energy approach.

Example 1.9

Determine the displacements of nodes of the spring system (Fig. 1.14).

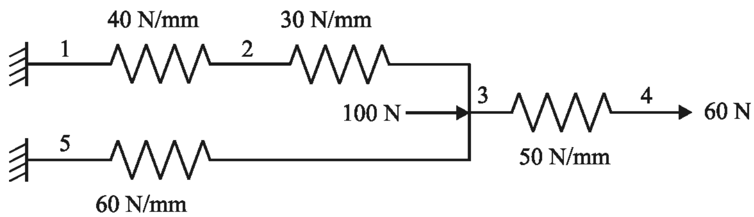


FIGURE 1.14 Example of a 4-noded spring system

Solution

Total potential energy of the system is given by

$$\pi = \left(\frac{1}{2}k_1\delta_1^2 + \frac{1}{2}k_2\delta_2^2 + \frac{1}{2}k_3\delta_3^2 + \frac{1}{2}k_4\delta_4^2\right) + \left(-F_3q_3 - F_4q_4\right)$$

where q_2, q_3, q_4 are the three unknown nodal displacements.

At the fixed points

$$q_1 = q_5 = 0$$

Extensions of the four springs are given by,

$$\delta_1 = q_2 - q_1 \;;\; \delta_2 = q_3 - q_2 \;;\; \delta_3 = q_4 - q_3 \;;\; \delta_4 = q_3 - q_5$$

For equilibrium of this 3-DOF system,

$$\frac{\partial \pi}{\partial q_i} = 0 \quad \text{for } i = 2,3,4$$

or $$\frac{\partial \pi}{\partial q_2} = k_1 q_2 - k_2(q_3 - q_2) = 0 \qquad \textbf{.....(a)}$$

$$\frac{\partial \pi}{\partial q_3} = k_2(q_3 - q_2) - k_3(q_4 - q_3) + k_4 q_3 - F_3 = 0 \qquad \textbf{.....(b)}$$

$$\frac{\partial \pi}{\partial q_4} = k_3(q_4 - q_3) - F_4 = 0 \qquad \textbf{.....(c)}$$

These equilibrium equations can be expressed in matrix form as

$$\begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 + k_4 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_3 \\ F_4 \end{Bmatrix}$$

or $$\begin{bmatrix} 40 + 30 & -30 & 0 \\ -30 & 30 + 50 + 60 & -50 \\ 0 & -50 & 50 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \\ 60 \end{Bmatrix}$$

Substituting

$$q_4 = \frac{F_4}{k_3} + q_3 = \left(\frac{60}{50}\right) + q_3 = 1.2 + q_3 \qquad \text{from eq. (c)}$$

and $$q_2 = \frac{k_2 q_3}{k_1 + k_2} = \frac{30 q_3}{(30 + 40)} = \frac{3 q_3}{7} \qquad \text{from eq. (a)}$$

in eq. (b), we get

$$k_2\left[q_3 - \frac{3q_3}{7}\right] - k_3\,[1.2 + q_3] + k_4 . q_3 - F_3 = 0$$

which gives, $q_3 = 2.0741$ mm
and then, $q_2 = 0.8889$ mm ; $q_4 = 3.2741$ mm

WHY FEM ?

The Rayleigh-Ritz method and potential energy approach are now of only academic interest. For a big problem, it is difficult to deal with a polynomial having as many coefficients as the number of DOF. FEM is a better generalization of these methods and extends beyond the discrete structures. Rayleigh-Ritz method of choosing a polynomial for displacement field and evaluating the coefficients for minimum potential energy is used in FEM, at the individual element level to obtain element stiffness matrix (representing load-displacement relations) and assembled to analyse the structure.

1.6 ORIGIN OF FEM

The subject was developed during 2^{nd} half of 20^{th} century by the contribution of many researchers. It is not possible to give chronological summary of their contributions here. Starting with application of force matrix method for swept wings by S. Levy in 1947, significant contributions by J.H.Argyris, H.L.Langhaar, R.Courant, M.J.Turner, R.W.Clough, R.J.Melosh, J.S.Przemieniecki, O.C.Zienkiewicz, J.L.Tocher, H.C.Martin, T.H.H.Pian, R.H.Gallaghar, J.T.Oden, C.A.Felippa, E.L.Wilson, K.J.Bathe, R.D.Cook etc... lead to the development of the method, various elements, numerical solution techniques, software development and new application areas.

Individual member method of analysis, being over-conservative, provides a design with bigger and heavier members than actually necessary. This method was followed in civil structures where weight is not a major constraint. Analysis of the complete structure was necessitated by the need for a better estimation of stresses in the design of airplanes with minimum factor of safety (and, hence, minimum weight), during World War-II. Finite element method, popular as FEM, was developed initially as Matrix method of structural analysis for discrete structures like trusses and frames.

FEM is also extended later for continuum structures to get better estimation of stresses and deflections even in components of variable cross-section as well as with non-homogeneous and non-isotropic materials, allowing for optimum design of complicated components. While matrix method was limited to a few discrete structures whose load-displacement relationships are derived from basic strength of materials approach, FEM was a generalisation of the method on the basis of variational principles and energy theorems and is applicable to all types of structures – discrete as well as continuum. It is based on conventional theory of elasticity (equilibrium of forces and Compatibility of displacements) and variational principles.

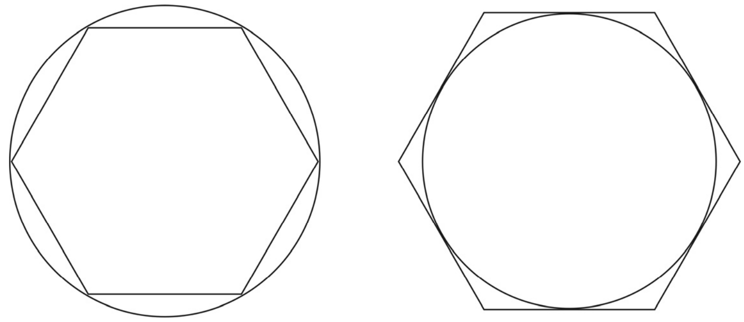
In FEM, the entire structure is analysed without using assumptions about the degree of fixity at the joints of members and hence better estimation of stresses in the members was possible. This method generates a large set of simultaneous equations, representing load-displacement relationships. Matrix notation is ideally suited for computerising various relations in this method. Development of numerical methods and availability of computers, therefore, helped growth of matrix method. Sound knowledge of strength of materials, theory of elasticity and matrix algebra are essential pre-requisites for understanding this subject.

1.7 PRINCIPLE OF FEM

In FEM, actual component is replaced by a simplified model, identified by a finite number of ***elements*** connected at common points called ***nodes***, with an assumed behaviour or response of each element to the set of applied loads, and evaluating the unknown field variable (displacement, temperature) at these finite number of points.

Example 1.10

The first use of this physical concept of representing a given domain as a collection of discrete parts is recorded in the evaluation of π from superscribed and inscribed polygons (Refer Fig. 1.15) for measuring circumference of a circle, thus approaching correct value from a higher value or a lower value (***Upper bounds/Lower bounds***) and improving accuracy as the number of sides of polygon increased (***convergence***). Value of π was obtained as 3.16 or $10^{1/2}$ by 1500 BC and as 3.1415926 by 480 AD, using this approach.



Case (a) Inscribed polygon Case (b) Superscribed polygon

FIGURE 1.15 Approximation of a circle by an inscribed and a superscribed polygon

Perimeter of a circle of diameter 10 cm = πD = 31.4 cm

Case-A : The circle of radius 'r' is now approximated by an inscribed regular polygon of side 's'. Then, using simple trigonometric concepts, the length of side 's' of any regular n-sided polygon can now be obtained as s = 2 r sin (360/2n). Actual measurements of sides of regular or irregular polygon inscribed in the circle were carried out in those days, in the absence of trigonometric formulae, to find out the perimeter.

With a 4-sided regular polygon, perimeter = 4 s = 28.284

With a 8-sided regular polygon, perimeter = 8 s = 30.615

With a 16-sided regular polygon, perimeter = 16 s = 31.215

approaching correct value from a lower value, as the number of sides of the inscribed polygon theoretically increases to infinity.

Case-B : The same circle is now approximated by a superscribed polygon of side 's', given by

$$s = 2\ r \tan(360/2n)$$

Then, With a 4-sided regular polygon, perimeter = 4 s = 40

With a 8-sided regular polygon, perimeter = 8 s = 33.137

With a 16-sided regular polygon, perimeter = 16 s = 31.826

approaching correct value from a higher value, as the number of sides of the circumscribed polygon theoretically increases to infinity.

A better estimate of the value of π (ratio of circumference to diameter) was found by taking average perimeter of inscribed and superscribed polygons, approaching correct value as the number of sides increases.

Thus with a 4-sided regular polygon, perimeter = (40+28.284) / 2 = 34.142

With a 8-sided regular polygon, perimeter = (33.137+30.615) / 2 = 31.876

With a 16-sided regular polygon, perimeter = (31.826+31.215) / 2 = 31.520

Example 1.11

In order to understand the principle of FEM, let us consider one more example, for which closed form solutions are available in every book of 'Strength of materials'. A common application for mechanical and civil engineers is the calculation of tip deflection of a cantilever beam AB of length 'L' and subjected to uniformly distributed load 'p'. For this simple case, closed form solution is obtained by integrating twice the differential equation.

$$EI\frac{d^2y}{dx^2} = M$$

and applying boundary conditions

$$y = 0 \text{ and } \frac{dy}{dx} = 0 \qquad \text{at } x = 0 \text{ (fixed end, A)},$$

we get, at x = L, $y_{max} = \dfrac{p L^4}{8 EI}$

This distributed load can be approximated as concentrated loads (P_1, P_2,…P_N) acting on 'N' number of small elements, which together form the total cantilever beam. Each of these concentrated loads is the total value of the distributed load over the length of each element ($P_1 = P_2 = \ldots = P_N = p\, L / N$), acting at its mid-point, as shown in Fig 1.16. Assuming that the tip deflection (at B) is small, the combined effect of all such loads can be obtained by linear superposition of the effects of each one of them acting independently. We will again make use of closed form solutions for the tip deflection values of a cantilever beam subjected to concentrated loads at some intermediate points.

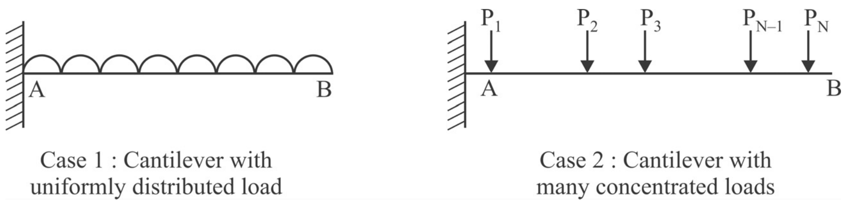


FIGURE 1.16 Cantilever beam with distributed load approximated by many concentrated loads

Tip deflection of the cantilever when subjected to concentrated load P_J at a distance L_J from the fixed end is given by

$$y_B = (y)_J + \left(\frac{dy}{dx}\right)_J .(L - L_J)$$

Closed form solutions for $(y)_J$ and $(dy/dx)_J$ can be obtained by integrating the beam deflection equation with appropriate boundary conditions, as

$$y_J = P_J \frac{L_J^4}{3\, EI}$$

$$\left(\frac{dy}{dx}\right)_J = P_J \frac{L_J^3}{2\, EI}$$

Deflection at B, y_B, due to the combined effect of all the concentrated loads along the length of the cantilever can now be obtained, by linear superposition, as

$$y_B = [(y)_1 + (dy/dx)_1 (L - L_1)] + [(y)_2 + (dy/dx)_2 (L - L_2)] + \ldots + [(y)_N + (dy/dx)_N (L - L_N)]$$

The results obtained with different number of elements are given in the table below, for the cantilever of length 200 cm, distributed load of 50 N/cm and EI = 10^9 N cm^2.

S. No.	No. of elements	Tip displacement, y_B (cm)
1	3	9.815
2	4	9.896
3	5	9.933
4	6	9.954
5	8	9.974
6	10	9.983
7	15	9.993
8	20	9.996

The exact value obtained for the cantilever with uniformly distributed load, from the closed form solution, is y_B = 10.0 cm. It can be seen, even in this simple case, that the tip deflection value approaches true solution from a lower value as the number of elements increases. In other words, the tip deflection value even with a small number of elements gives an approximate solution.

This method in this form is not useful for engineering analysis as the approximate solution is lower than the exact value and, in the absence of error estimate, the solution is not practically useful.

FEM approach, based on minimum potential energy theorem, ***converges to the correct solution from a higher value*** as the number of elements in the model increases. While the number of elements used in a model is selected by the engineer, based on the required accuracy of solution as well as the availability of computer with sufficient memory, FEM has become popular as it ensures usefulness of the results obtained (on a more conservative side) even with lesser number of elements.

Finite Element Analysis (FEA) based on FEM is a simulation, not reality, applied to the mathematical model. Even very accurate FEA may not be good enough, if the mathematical model is inappropriate or inadequate. A mathematical model is an idealisation in which geometry, material properties, loads and/or boundary conditions are simplified based on the analyst's understanding of what features are important or unimportant in obtaining the results required. The error in solution can result from three different sources.

Modelling error – associated with the approximations made to the real problem.

Discretisation error – associated with type, size and shape of finite elements used to represent the mathematical model; can be reduced by modifying mesh.

Numerical error – based on the algorithm used and the finite precision of numbers used to represent data in the computer; most softwares use double precision for reducing numerical error.

It is entirely possible for an unprepared software user to misunderstand the problem, prepare the wrong mathematical model, discretise it inappropriately, fail to check computed output and yet accept nonsensical results. FEA is a solution technique that removes many limitations of classical solution techniques; but does not bypass the underlying theory or the need to devise a satisfactory model. Thus, ***the accuracy of FEA depends on the knowledge of the analyst in modelling the problem correctly***.

1.8 CLASSIFICATION OF FEM

The basic problem in any engineering design is to evaluate displacements, stresses and strains in any given structure under different loads and boundary conditions. Several approaches of Finite Element Analysis have been developed to meet the needs of specific applications. The common methods are :

Displacement method – Here the structure is subjected to applied loads and/or specified displacements. The primary unknowns are displacements, obtained by inversion of the stiffness matrix, and the derived unknowns are stresses and strains. Stiffness matrix for any element can be obtained by variational principle, based on minimum potential energy of any stable structure and, hence, this is the most commonly used method.

Force method – Here the structure is subjected to applied loads and/or specified displacements. The primary unknowns are member forces, obtained by inversion of the flexibility matrix, and the derived unknowns are stresses and strains. Calculation of flexibility matrix is possible only for discrete structural elements (such as trusses, beams and piping) and hence, this method is limited in the early analyses of discrete structures and in piping analysis

Mixed method – Here the structure is subjected to applied loads and/or specified displacements. The method deals with large stiffness coefficients as well as very small flexibility coefficients in the same matrix. Analysis by this method leads to numerical errors and is not possible except in some very special cases.

Hybrid method – Here the structure is subjected to applied loads and stress boundary conditions. This deals with special cases, such as airplane door frame which should be designed for stress-free boundary, so that the door can be opened during flight, in cases of emergencies.

Displacement method is the most common method and is suitable for solving most of the engineering problems. The discussion in the remaining chapters is confined to displacement method.

1.9 TYPES OF ANALYSES

Mechanical engineers deal with two basic types of analyses for discrete and continuum structures, excluding other application areas like fluid flow, electromagnetics. FEM helps in modelling the component once and perform both the types of analysis using the same model.

(a) ***Thermal analysis*** – Deals with steady state or transient heat transfer by conduction and convection, both being linear operations while radiation is a non-linear operation, and estimation of temperature distribution in the component. This result can form one part of load condition along with internal pressure etc., for the structural analysis.

(b) ***Structural analysis*** – Deals with estimation of stresses and displacements in discrete as well as continuum structures under various types of loads such as gravity, wind, pressure and temperature. Dynamic loads may also be considered.

1.10 BASIC STEPS IN FEA

Basic steps in the analysis by Finite Element Method are listed out for the general case of structural analysis for static loads. The details in these steps will vary for other types of analysis, such as thermal analysis, dynamic analysis, etc.

(a) Creating a ***mathematical model*** from Physical model – Identifying structure (1-D / 2-D / 3-D) based on relative dimensions and neglecting insignificant details such as rivets / bolts, pins etc.; Identifying loads (axial / normal) and expected behaviour of the structure (axial / in-plane / bending deformation); Identifying material properties (isotropic / orthotropic / elastic / plastic / linear / non-linear / constant / temperature-dependent)

(b) Creating a ***Finite element model*** by identifying appropriate types of elements to represent the behaviour of the structure and discretising the structure into a finite number of elements and nodes, minimising the model by taking advantage of symmetry wherever applicable; specifying material properties and section properties for all the elements

(c) Calculating stiffness matrix and transformation matrix for transformation from local to global coordinate system, for each element

(d) Assembling element stiffness matrices and storing banded matrix, taking advantage of symmetry and banded nature, to minimise memory requirement

(e) Applying specified boundary conditions, by eliminating corresponding rows and columns

(f) Assembling load vector, using consistent nodal loads in place of distributed loads

(g) Calculating primary unknowns (nodal displacements in global coordinate system), by inversion of the reduced stiffness matrix

(h) Calculating secondary unknowns (such as stresses and strains in each element) in local coordinate system of each element

(i) Calculation of principal stresses, principal strains etc for validation of design. For visual check of results, plots of deformed geometry, stress contours can also be obtained.

1.11 SUMMARY

- Finite Element Method, popularly known as FEM, involves analysis of the entire structure, instead of separately considering individual elements with simplified or assumed end conditions. It thus helps in a more accurate estimate of the stresses in the members, facilitating optimum design.
- FEM involves idealizing the given component into a finite number of small elements, connected at nodes. FEM is an extension of Rayleigh-Ritz method, eliminating the difficulty of dealing with a large polynomial representing a suitable displacement field valid over the entire structure. Over each finite element, the physical process is approximated by functions of desired type and algebraic equations, which relate physical quantities at these nodes and are developed using variational approach. Assembling these element relationships in the proper way is assumed to approximately represent relationships of physical quantities of the entire structure.
- FEM is based on minimum potential energy theorem. It approaches true solution from a higher value, as the number of elements increases. Thus, it gives a conservative solution even with a small number of elements, representing a crude idealisation.

OBJECTIVE QUESTIONS

1. The solution by FEM is
   (a) always exact (b) mostly approximate
   (c) sometimes exact (d) never exact
2. Discrete analysis covers
   (a) all 2-D trusses & frames (b) all 3-D trusses & frames
   (c) all 2-D and 3-D trusses & frames (d) no trusses; only frames
3. FEM is a generalization of
   (a) Rayleigh-Ritz method (b) Weighted residual method
   (c) Finite difference method (d) Finite volume method
4. Variational principle is the basis for
   (a) Displacement method (b) Weighted residual method
   (c) Finite difference method (d) Finite volume method
5. Displacement method is based on minimum
   (a) potential energy
   (b) strain energy
   (c) complementary strain energy
   (d) work done
6. Hybrid method is best suited for problems with prescribed
   (a) displacements (b) forces (c) stresses (d) temperature
7. Primary variable in FEM structural analysis is
   (a) displacement (b) force (c) stress (d) strain
8. Stress boundary conditions can be prescribed in
   (a) displacement method (b) hybrid method
   (c) force method (d) mixed method
9. Prescribed loads can form input data in
   (a) displacement method (b) hybrid method
   (c) force method (d) mixed method
10. Stiffness matrix approach is used in
    (a) displacement method (b) stress method
    (c) force method (d) mixed method
11. Displacement method of FEM for structural analysis gives
    (a) stiffness matrix (b) flexibility matrix
    (c) conductance matrix (d) mixed matrix

12. Flexibility matrix approach is used in
    (a) displacement method
    (b) stress method
    (c) force method
    (d) mixed method