1. Basic Governing Equations

1.1 Introduction

Many of the atmospheric motion systems are driven by differential heating. Examples of this are the atmospheric general circulation as also some weather systems. In particular, tropical cyclones are driven by the release of latent heat in the central region. Thus we need equations of fluid dynamics and thermodynamics for the study of such systems.

The basic governing equations of atmospheric motion are:

(i) Equation of motion
(ii) Continuity equation
(iii) Equation of state
(iv) First law of thermodynamics

1.2 Equation of Motion

This is a statement of Newton's second law of motion for a parcel of air (of unit mass) in the atmosphere. It is written as

$$\frac{d\vec{V}}{dt} = -\alpha\nabla p + \vec{g} - 2\vec{\Omega} \times \vec{V} + \vec{F}$$

where p is pressure, $\alpha = \frac{1}{\rho}$, ρ is density, $\vec{V}$ is vector wind, $\vec{g}$ is gravity, $\vec{\Omega}$ is angular velocity of earth, $\vec{F}$ is frictional force.

The acceleration is equal to the vector sum of all forces (per unit mass, as mentioned above) acting on the parcel, such as

(i) Pressure gradient force[1]
(ii) Gravity
(iii) Coriolis force
(iv) Frictional force

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[1]Appendix 1

The coriolis force arises from the rotation of the earth. We begin with a non-rotating co-ordinate system fixed at the centre of the earth and then transform to a rotating co-ordinate system[2].

It is more convenient to decompose this vector equation into its three scalar components in a local Cartesian system with the *x*-axis pointing to the east, *y*-axis pointing to the north and vertical axis pointing upwards.

The component scalar equations are

$$\frac{du}{dt} = -\alpha\frac{\partial p}{\partial x} + fv + F_x$$
$$\frac{dv}{dt} = -\alpha\frac{\partial p}{\partial y} - fu + F_y$$
$$\frac{dw}{dt} = -\alpha\frac{\partial p}{\partial z} - g + F_z$$

where u, v, w are the scalar components of the wind $\vec{V}$ along the x, y and z co-ordinates and $f = 2\Omega\sin\phi$ is known as the coriolis parameter. ϕ is the latitude.

In the vertical equation of motion, for large-scale motion, the vertical acceleration is usually small and there is a very good balance between gravity and the vertical pressure gradient force. This is known as hydrostatic balance.

$$-\alpha\frac{\partial p}{\partial z} = g$$

The total derivative $\frac{du}{dt}$ is a derivative following the fluid parcel.

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

where $\frac{\partial u}{\partial t}$ is the local derivative at a location and $u\frac{\partial u}{\partial x}$, $v\frac{\partial u}{\partial y}$ and $w\frac{\partial u}{\partial z}$ are known as advection terms.

[2]Appendix 2

1.3 Continuity Equation

This is a statement of the physical principle of conservation of mass.

Consider a fixed volume *dx dy dz.*

Rate of inflow of fluid at wall $A = \rho u\, dy\, dz$
(Fig. 1.1)

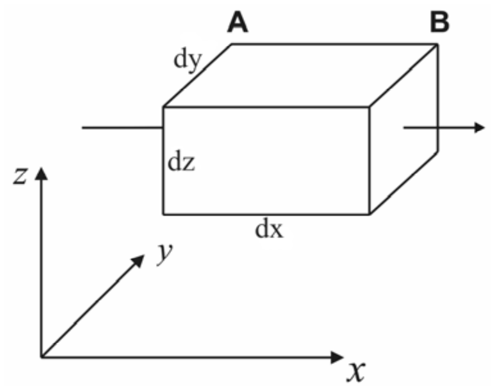

Fig. 1.1

At wall *B* it is $-\left\{\rho u + \frac{\partial}{\partial x}(\rho u)\,dx\right\} dy\ dz$

It is an outflow.

$$\text{Net inflow} = \rho u\, dy\, dz - \left(\rho u + \frac{\partial}{\partial x}(\rho u)\,dx\right) dy\, dz$$

$$= -\frac{\partial}{\partial x}(\rho u)\,dx\ dy\ dz$$

Similarly, inflow at all the six walls

$$= -\left\{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right\} dx\, dy\, dz$$

This is equal to the local rate of change of mass $\frac{\partial}{\partial t}(\rho\ dx\, dy\, dz)$.

Therefore,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \left(\rho \vec{V}\right)$$

$$= -\vec{V} \cdot \nabla \rho - \rho \nabla \cdot \vec{V}$$

i.e.,
$$\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{V}$$

or
$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \vec{V}$$

1.4 Equation of State

By combining Charles and Boyle's laws, we get the equation of state

$pV = nR_u T$ for n moles.

For unit mass it is $p = \rho R T$

or $p\alpha = RT$ where R (gas constant) is for unit mass.

For the temperature range we encounter in the massive part of the atmosphere, we can treat the atmosphere as a perfect gas (mixture of perfect gases). See Appendix 3.

1.5 First Law of Thermodynamics

If we add a small quantity of heat dQ to a fluid, part of it goes to increase the internal energy (dU) and part of it goes to do work (dW).

Thus, $dQ = dU + dW$

$dQ = dU + p\,d\alpha$ for unit mass.

If heat is added at constant volume $d\alpha = 0$.

Then $dQ = dU = C_v dT$

Therefore, in general $dQ = C_v\,dT + p\,d\alpha$

Now, $p\alpha = RT$

Differentiating

$$p\,d\alpha + \alpha\,dp = R\,dT$$

Therefore, $dQ = C_v\,dT + R\,dT - \alpha\,dp$

Now, if we add heat at constant pressure $dp = 0$

Therefore, $dQ = C_p\,dT = \left(C_v + R\right)dT$

Therefore in general,

$$dQ = C_p\,dT - \alpha\,dp$$

Writing these as differential equations

$$\begin{aligned}\frac{dQ}{dt} &= C_v\frac{dT}{dt} + p\frac{d\alpha}{dt}\\ &= C_p\frac{dT}{dt} - \alpha\frac{dp}{dt}\end{aligned}$$

It is convenient to define a new variable called potential temperature.

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa}$$

where $\kappa = \dfrac{R}{C_p}$ and p_0 is a reference pressure usually taken as 1000 hpa (or 1000 mb).

The potential temperature θ of a parcel at temperature T and pressure p is defined as the temperature attained by it when it is adiabatically brought (compression or expansion) to 1000 hpa. Adiabatic process is one in which heat is neither added nor taken out i.e., $dQ = 0$.

1.6 Co-ordinate Systems

The most commonly used co-ordinate system is the local co-ordinate system. This is a right-handed Cartesian co-ordinate system on a local tangential plane, which is fixed to the earth at the local point of consideration. *x*-axis points to the east. Distance towards the east is positive and to the west is negative. *y*-axis points to the north. Distance towards north is positive and towards south is negative. *z*-axis points upwards, distance upwards is positive and downwards is negative. (Fig. A2.2)

$\hat{i}, \hat{j}, \hat{k}$ are unit vectors.

The position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w$$

The velocity (vector)

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

Acceleration,

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k}$$

Now,
$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$\frac{\partial u}{\partial t}$ is known as the local derivative or local change and $u\frac{\partial u}{\partial x}, v\frac{\partial u}{\partial y}$ are known as the horizontal advection terms and $w\frac{\partial u}{\partial z}$ is known as the vertical advection term.

The physical significance of these terms needs to be understood clearly. It is easier to understand if we use the variable, temperature *T*.

$\frac{\partial T}{\partial t}$ is the local change of temperature. If we measure temperature at one location, say the meteorological observatory at Bangalore and note the rate of change of temperature with respect to time at this location, it is known as the local change.

If we have a situation such that isotherms (line connecting places of same temperature) run north-south as shown in the diagram and we have a westerly wind (wind blowing from the west) then air is moving from colder region to a warmer region. This is a case of cold advection. Fig. 1.2(a)

If we have isotherms and wind as in the next diagram, we have a case of warm advection. Fig. 1.2(b).

If we have isotherms running east-west and a southerly wind (wind blowing from the south) as shown in the next diagram, we have cold advection. Fig. 1.2(c).

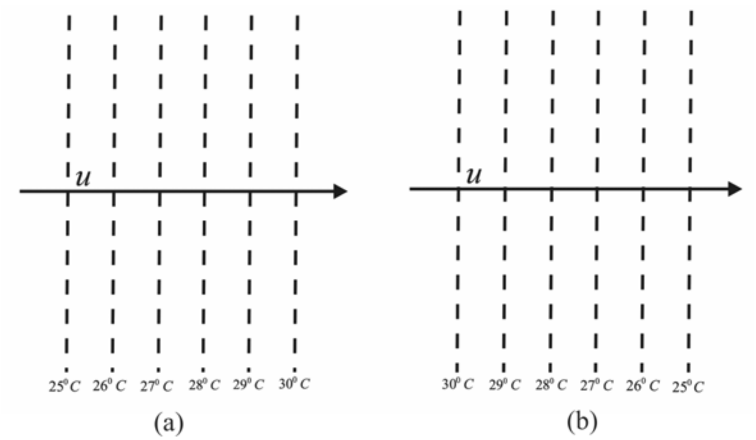

Fig. 1.2 Horizontal advection

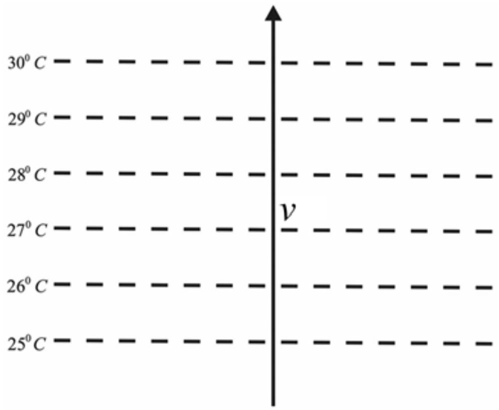

Fig. 1.2c Horizontal advection

Thermal advection is very important in the middle and higher latitudes (extra-tropics). When there are strong northerly winds blowing from Canada towards USA during (northern) winter, it is a case of cold advection. It is sometimes referred to as the Canadian express.

In Asia if there are strong northerly winds (winds blowing from the north) blowing from Siberia in winter, it is a case of cold advection. It can be referred to as the Siberian express. India is protected from these cold winds by the mighty Himalayas.

In the tropics thermal advection is less pronounced. Here moisture advection is prominent. There is large moisture advection by the monsoon south-westerly and southerly winds from the Arabian sea and the Indian ocean.

1.7 Spherical Polar Co-ordinates

As the earth is almost a sphere it is most natural to use spherical polar co-ordinates. The co-ordinates are λ the longitude, ϕ the latitude and z the vertical distance above the surface. (Fig. 1.3).

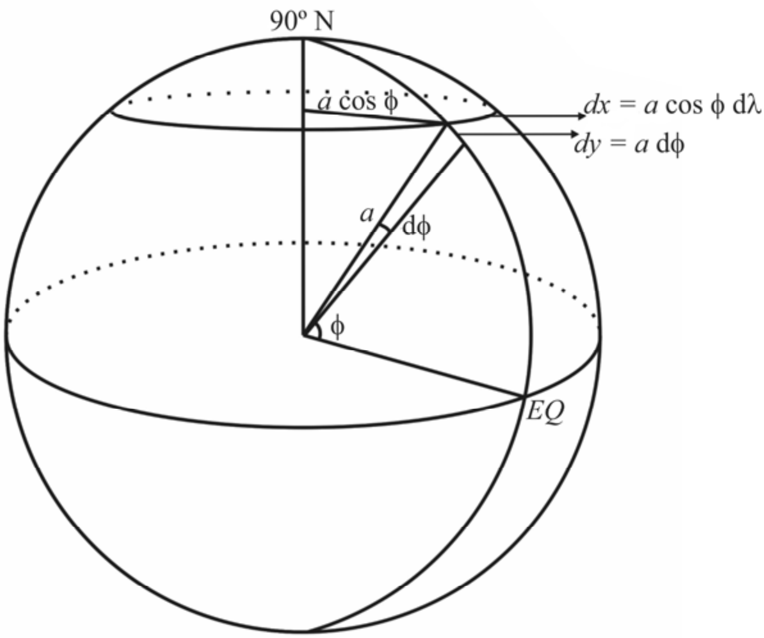

Fig. 1.3 Spherical polar co-ordinates

Now $dx = a\cos\phi\, d\lambda$ and $dy = a\, d\phi$

The velocity is

$$\vec{V} = \hat{i}\,u + \hat{j}\,v + \hat{k}\,w$$

where $u = a\cos\phi \dfrac{d\lambda}{dt}$, $v = a\dfrac{d\phi}{dt}$ and $w = \dfrac{dz}{dt}$

$$r = a + z \approx a$$

where a = radius of the earth.

It is to be noted that the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are functions of the position on the earth.

Taking this into consideration, the component equations are

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\alpha\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_x$$
$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\alpha\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + F_y$$
$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\alpha\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + F_z$$

For synoptic and large scale motions, the horizontal length scales are of the order of 1000 km, time scale is of the order of a day (10^5 second), vertical height scale is of the order of 10 km and horizontal wind speed is of the order of 10 ms^{-1}. However, the vertical wind speed is much smaller and is of the order of 1 cms^{-1}.

For these scales (or small Rossby number[3] regime), the equations of motion take the simpler form

$$\frac{du}{dt} = -\alpha\frac{\partial p}{\partial x} + fv + F_x$$
$$\frac{dv}{dt} = -\alpha\frac{\partial p}{\partial y} - fu + F_y$$
$$\frac{dw}{dt} = -\alpha\frac{\partial p}{\partial z} - g + F_z$$

[3]Appendix 4

For large scale motion, the accelerations are one order of magnitude smaller. So,

$$\left.\begin{aligned} +\alpha\frac{\partial p}{\partial x} &\approx fv \qquad & v_g &= \frac{\alpha}{f}\frac{\partial p}{\partial x} \\ +\alpha\frac{\partial p}{\partial y} &\approx -fu \qquad & u_g &= -\frac{\alpha}{f}\frac{\partial p}{\partial y} \end{aligned}\right\}$$

This is known as geostrophic balance and $\vec{V}_g = u_g \hat{i} + v_g \hat{j}$ is the geostrophic wind.

In vector notation

$$-\alpha \nabla_H p - f\hat{k} \times \vec{V}_g = 0$$

i.e.,
$$\vec{V}_g = -\frac{\alpha}{f} \nabla_H \, p \times \hat{k}$$

Large scale wind blows parallel to lines of constant pressure or isobars. See Fig. 1.5c.

During the monsoon season the pressure gradient over peninsular India is from south to north. Therefore the zonal wind is westerly (with low pressure to the left in the northern hemisphere).

In addition, we have, of course, the hydrostatic balance i.e.,

$$\alpha\frac{\partial p}{\partial z} = -g$$

1.8 Pressure Co-ordinates

Now, $dp = -g\rho\, dz$

As the large scale atmosphere is in good hydrostatic balance it is possible to use pressure as a vertical co-ordinate (independent variable) instead of height *z*. Then *z*, the height of constant pressure surface or $\phi = gz$, the geopotential becomes the dependent variable.

In, x, y, p – co-ordinates; $\dot{x} = u$, $\dot{y} = v$, $\omega = \dfrac{dp}{dt}$.

The total derivatives is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p}$$

The equations of motion in this co-ordinate system are

$$\frac{du}{dt} = -\frac{\partial \phi}{\partial x} + f\,v + F_x$$
$$\frac{dv}{dt} = -\frac{\partial \phi}{\partial y} - f\,u + F_y$$
$$-g\frac{\partial z}{\partial p} = -\frac{\partial \phi}{\partial p} = \frac{1}{\rho} = \alpha \quad \text{or} \quad \alpha = -\frac{\partial \phi}{\partial p}$$

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Thus it becomes simpler.

1.9 Natural Co-ordinates

We consider a unit vector $\hat{t}$ parallel to the horizontal wind vector (or streamline) and $\hat{n}$ perpendicular to the wind, positive to the left of the flow direction.

$\hat{k}$ is directed vertically upwards.

$$\vec{V} = V\,\hat{t}$$

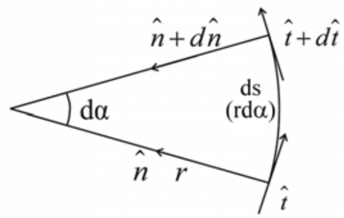

Fig. 1.4 Natural coordinates

$$\frac{d\vec{V}}{dt} = \frac{dV}{dt}\hat{t} + V\frac{d\hat{t}}{dt}$$

From the figure 1.4, $d\hat{t} = d\alpha\hat{n}$

$$\begin{aligned}\frac{d\vec{V}}{dt} &= \dot{V}\hat{t} + V\frac{d\alpha}{dt}\hat{n} \\ &= \dot{V}\hat{t} + V\frac{1}{r}\frac{r\,d\alpha}{dt}\hat{n}\end{aligned}$$

where r = radius of curvature

$$= \dot{V}\hat{t} + \frac{V^2}{r}\hat{n}$$

$\dot{V} = \dfrac{dV}{dt}$ is the tangential acceleration

$\dfrac{V^2}{r}$ is the centrifugal acceleration

Since the Coriolis force acts always normal to the direction of motion

$$-f\hat{k} \times \vec{V} = -fV\hat{n}$$

The equations of motion in natural co-ordinates are

$$\begin{aligned}\frac{dV}{dt} &= -\alpha\frac{\partial p}{\partial s} \\ \frac{V^2}{r} + fV &= -\alpha\frac{\partial p}{\partial n}\end{aligned}$$

Along the direction of motion, the tangential pressure gradient accelerates the wind. In the normal direction, the pressure gradient force is balanced by Coriolis force and centrifugal force.

1.10 Gradient Wind

Here the flow (wind) is parallel to isobars and is called gradient wind. (Fig. 1.5 a, b)

The tangential acceleration,

$$\frac{dV}{dt} = 0$$

$$\frac{V^2}{r} + fV = -\alpha \frac{\partial p}{\partial n}$$

There is a balance between pressure gradient force, coriolis force and centrifugal force.

In a low pressure area (Fig. 1.5a) the flow is anticlockwise in the northern hemisphere. In regions of high pressure the flow is clockwise. Regions of low pressure or cyclonic circulation in the lower levels are usually associated with weather. An exception is the shallow heat low over northwest India-Pakistan during the monsoon season. Regions of high pressure or anticyclones (Fig. 1.5b) in the lower levels are usually associated with sinking and clear weather.

If $r \to \infty$

The flow is in a straight line

$$fV_g = -\alpha \frac{\partial p}{\partial n}$$

This is geostrophic wind.

There is a balance between pressure gradient force and coriolis force.
(Fig. 1.5 c).

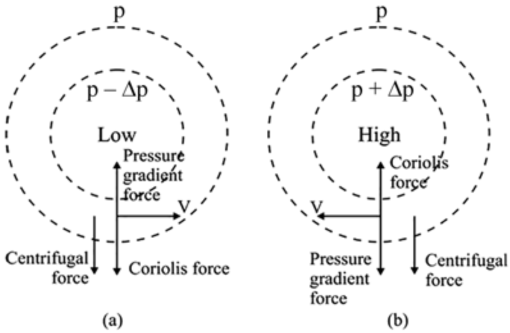

Fig. 1.5 (a & b) Gradient wind

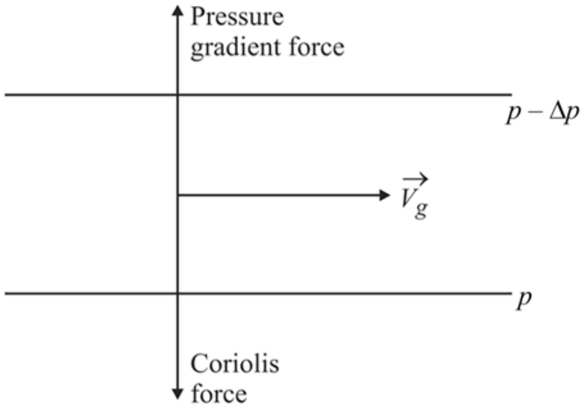

Fig. 1.5 (c) Geostrophic wind

1.11 Thermal Wind-variation of Wind in the Vertical

The geotrophic wind $\vec{V}_g = -\dfrac{g}{f}\nabla_p z \times \hat{k}$

In the vertical $\Delta_z \vec{V}_g = -\dfrac{g}{f}\nabla_p (\Delta z) \times \hat{k}$

$$dp = -g\rho\, dz$$
$$\alpha\, dp = -g\, dz$$
$$p\alpha = RT$$
$$RT\frac{dp}{p} = -g\, dz = -d\phi$$
$$dz = -\frac{RT}{g} d\ln p$$
$$\Delta z = -\frac{R\bar{T}}{g}\Delta(\ln p)$$
$$= \frac{R\bar{T}}{g}\ln\left(\frac{p_1}{p_2}\right)$$

Thermal wind $\Delta\vec{V}_g = -\dfrac{R}{f}\ln\left(\dfrac{p_1}{p_2}\right)\nabla_p \bar{T} \times \hat{k}$

The thermal wind blows parallel to isotherms with low temperature to the left (in the northern hemisphere). During monsoon the thermal wind is easterly.

1.12 Vorticity and Divergence

There are two quantities derived from the wind $\vec{V}$, which are very useful.

Vorticity $\qquad \nabla \times \vec{V} = \xi \hat{i} + \eta \hat{j} + \zeta \hat{k}$

The vertical component of vorticity is

$$\zeta = \hat{k} \cdot \nabla \times \vec{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The circulation around *ABCDA* (Fig. 1.6)

$$= \oint \vec{V} \ . \ dl$$

$$dC = \left(u - \frac{\partial u}{\partial y}\frac{dy}{2} \right) dx + \left(v + \frac{\partial v}{\partial x}\frac{dx}{2} \right) dy$$

$$- \left(u + \frac{\partial u}{\partial y}\frac{dy}{2} \right) dx - \left(v - \frac{\partial v}{\partial x}\frac{dx}{2} \right) dy$$

$$= u\,dx - \frac{\partial u}{\partial y}\frac{dy}{2} dx + v\,dy + \frac{\partial v}{\partial x}\frac{dx}{2} dy$$

$$- u\,dx - \frac{\partial u}{\partial y}\frac{dy}{2} dx - v\,dy + \frac{\partial v}{\partial x}\frac{dx}{2} dy$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$$

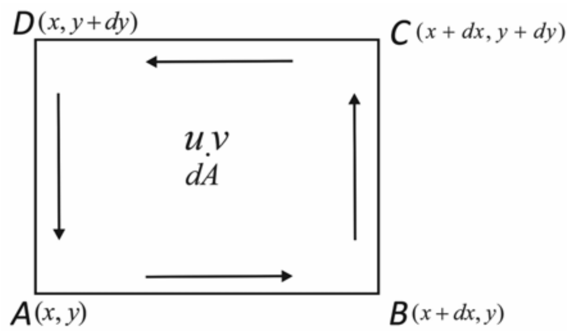

Fig. 1.6 Circulation and vorticity

The circulation per unit area

$$\zeta = \frac{dC}{dA} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Vorticity shows rotation,

Divergence

$$D = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial y} \right) \cdot \left(\hat{i}\,u + \hat{j}\,v + \hat{k}\,w \right)$$

$$= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

Horizontal divergence

$$D_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Now the horizontal wind can be expressed as

$$\vec{V} = \hat{k} \times \nabla\psi + \nabla\chi$$

$$= \vec{V}_\psi + \vec{V}_\chi$$

where, ψ - the stream function
χ - velocity potential

$$\vec{V}_\psi = \hat{k} \times \left(\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} \right)$$

$$= \hat{j}\,\frac{\partial\psi}{\partial x} - \hat{i}\,\frac{\partial\psi}{\partial y}$$

$$u_\psi = -\frac{\partial\psi}{\partial y}\,; \quad v_\psi = \frac{\partial\psi}{\partial x}$$

Similarly,

$$\vec{V}_\chi = \hat{i}\frac{\partial\chi}{\partial x} + \hat{j}\frac{\partial\chi}{\partial y}$$

$$u_\chi = \frac{\partial\chi}{\partial x}\,; \quad v_\chi = \frac{\partial\chi}{\partial y}$$

Therefore,

$$
\begin{aligned}
u &= u_\psi + u_\chi \\
&= -\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x} \\
v &= \frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y}
\end{aligned}
$$

The vertical component of vorticity

$$
\begin{aligned}
\zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial y}\right) \\
&= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi
\end{aligned}
$$

Only ψ component (the rotational part of wind) contributes to vorticity.

The horizontal divergence

$$
\begin{aligned}
D_H &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y}\right) \\
&= \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = \nabla^2 \chi
\end{aligned}
$$

Only χ component of wind contributes to divergence.

1.13 Vorticity Equation

From the two component equations of motion along the x (longitude) and y (latitude) directions, it is convenient to obtain a derived equation known as the vorticity equation.

$$
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\alpha\frac{\partial p}{\partial x} + f\,v + F_x \qquad \ldots (1.1)
$$

$$
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\alpha\frac{\partial p}{\partial y} - f\,u + F_y \qquad \ldots (1.2)
$$

Taking $\dfrac{\partial}{\partial y}$ of equation (1.1) and subtracting from $\dfrac{\partial}{\partial x}$ of equation (1.2)

$$-\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+u\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)+\right.$$
$$\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}+w\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)=-\frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}-\alpha\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)+f\frac{\partial v}{\partial y}+$$
$$\left.\beta v+\frac{\partial F_x}{\partial y}\right] \qquad \ldots (1.3)$$

$$+\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right)+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\right.$$
$$\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}+w\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right)=-\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y}-\alpha\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)-f\frac{\partial u}{\partial x}+$$
$$\left.\frac{\partial F_y}{\partial x}\right] \qquad \ldots (1.4)$$

Regrouping and adding terms of similar type

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+$$
$$w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+$$
$$\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)=-\left(\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}\right)-f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-$$
$$\beta v+\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}$$

i.e.,

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial z}+w\frac{\partial \zeta}{\partial z}+\beta v$$

$$=-(\zeta+f)D_H+\left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right)+\left(\frac{\partial p}{\partial x}\frac{\partial \alpha}{\partial y}-\frac{\partial p}{\partial y}\frac{\partial \alpha}{\partial x}\right)+$$
$$\left(\frac{\partial F_x}{\partial x}-\frac{\partial F_y}{\partial y}\right)$$

i.e.,
$$\frac{d}{dt}(\zeta+f)=-(\zeta+f)D_H+\left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right)+\left(\frac{\partial p}{\partial x}\frac{\partial \alpha}{\partial y}-\frac{\partial p}{\partial y}\frac{\partial \alpha}{\partial x}\right)+\left(\frac{\partial F_x}{\partial x}-\frac{\partial F_y}{\partial y}\right)$$

The absolute vorticity of a fluid parcel can change by (i) divergence term, (ii) tilting term, (iii) solenoidal term and, (iv) friction term.

The largest term is the divergence term. In regions of horizontal convergence (D_H negative), the vorticity increases.

The second term is called the vortex tube term. This term converts horizontal component of vorticity into the vertical component.

The third term is called the solenoidal term. If iso-lines of α (or ρ) and iso-lines of pressure or isobars intersect at an angle, we have pressure-specific volume solenoids.

As an approximation

$$\frac{d\zeta_a}{dt}\approx-\zeta_a D_H \quad \text{or} \quad \frac{dl_n}{dt}\zeta_a\approx-D_H=-\nabla_H\cdot\vec{V}$$

If the divergence is zero, as in a nondivergent barotropic atmosphere, then

$$\frac{d\zeta_a}{dt}=0$$

Thus, in this case the absolute vorticity is conserved.

1.14 Humidity

The atmosphere contains a variable amount of moisture. Water occurs in all three phases – ice/snow, water and vapour.

Water vapour is a very important constituent of the atmosphere. Obviously, it is an essential ingredient for cloud and rain formation. It also absorbs terrestrial infrared radiation and keeps the earth warm.

When water vapour condenses to water the released latent heat is an important secondary source of heating for tropical and monsoon disturbances as well as for the maintenance of monsoon circulation and the Hardley circulation.

There is always evaporation from the oceans and other water bodies. The amount of evaporation depends upon wind speed and the vertical gradient of water vapour above the water surface.

The partial pressure of water vapour is referred to as vapour pressure, e. Under equilibrium conditions, when the maximum possible water vapour has evaporated, the atmosphere is said to be saturated. Then the vapour pressure is referred to as saturation vapour pressure, e_s. The mixture of air and water vapour, when it is saturated is referred to as saturated air. When vapour pressure is less than saturation vapour pressure, air is said to be unsaturated. When saturated air is cooled – as it happens when air rises:

(i) when air flows over a mountain barrier or
(ii) in regions of low level convergence as in low pressure areas, depressions or storms, the saturation vapour pressure at the new lower temperature is less. The atmosphere can hold less amount of water. The excess water vapour usually condenses.

The latent heat of vaporization is the amount of heat required to change unit mass (one gram or kilogram) of water into water vapour at the same temperature. On the other hand when one gram or kilogram of water vapour condenses to water, latent heat is released.

The equation of state for water vapour is

$$e = \rho_v R_v T$$

where R_v is gas constant of water vapour.

Clausius-Clapeyron equation

$$\frac{1}{e_s}\frac{de_s}{dT} = \frac{L_{lv}}{R_v T^2}$$

This tells us about the variation of saturation vapour pressure with temperature. Saturation vapour pressure increases with temperature. At higher temperatures as in the tropics, saturation vapour pressure is higher. The atmosphere can hold more moisture.

As said earlier moist air is a mixture of dry air and water vapour. There are different ways of expressing the moisture content in the atmosphere.

Mixing ratio is the mass of water vapour per unit mass of dry air in the mixture.

$$m = \frac{M_v}{M_d} = \frac{\frac{M_v}{V}}{\frac{M_d}{V}} = \frac{\rho_v}{\rho_d}$$

Specific humidity q is the mass of water vapour per unit mass of (moist) air.

$$q = \frac{M_v}{M} = \frac{M_v}{M_d + M_v} = \frac{\rho_v}{\rho_d + \rho_v}$$

m and q are dimensionless. They are expressed in gram per kilogram.

Relative humidity is the ratio of the observed mixing ratio and the saturation mixing ratio.

$$r = \frac{m}{m_s} \approx \frac{e}{e_s}$$

Relative humidity is generally expressed as a percentage.

1.15 Hydrostatic Stability

When a small parcel of air is displaced vertically (without any mixing), if it comes back to its original position, then the atmosphere is said to be in hydrostatic equilibrium. If the parcel moves away, the atmosphere is unstable. If the parcel rests in any position it is moved to, it is in neutral equilibrium.

The surrounding air is in hydrostatic equilibrium

i.e., $$dp = -g\,\rho\,dz \qquad \dots (1.5)$$

The parcel (of density ρ') may not be in hydrostatic equilibrium.

$$\frac{dw}{dt} = -g - \frac{1}{\rho'}\left(\frac{\partial p}{\partial z}\right) \qquad \dots (1.6)$$

ρ' is the density of displaced parcel, $p' = p$ pressure adjusts quickly.

Eliminate $\frac{\partial p}{\partial z}$

$$\frac{dw}{dt} = -g + \frac{1}{\rho'} g\rho = g\left(\frac{\rho - \rho'}{\rho'}\right) \qquad \ldots (1.7)$$

$$= \frac{g(\gamma - \gamma')z}{T}$$

where $\gamma = -\frac{\partial T}{\partial z}$ is the lapse rate

If $\gamma > \gamma'$ Unstable

If $\gamma = \gamma'$ Neutral

If $\gamma < \gamma'$ Stable

The displacements are adiabatic.

In the atmosphere which contains moisture,

$\gamma < \gamma_s$ absolutely stable

$\gamma_s < \gamma < \gamma_d$ conditionally unstable, stable till the parcel is unsaturated, unstable when it becomes saturated.

$\gamma > \gamma_d$ absolutely unstable[4]

γ_d is dry adiabatic lapse rate (DALR) and γ_s is saturation adiabatic lapse rate (SALR).

Now, $$\theta = T\left(\frac{p_0}{p}\right)^{R/C_p}$$

$$\frac{\partial \ln \theta}{\partial z} = \frac{\partial \ln T}{\partial z} - \frac{R}{C_p}\frac{\partial \ln p}{\partial z}$$

[4]Appendix 5

$$\frac{1}{\theta}\frac{\partial\theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} - \frac{R}{C_p}\frac{1}{p}\frac{\partial p}{\partial z}$$

$$= \frac{1}{T}\frac{\partial T}{\partial z} + \frac{R}{C_p}\frac{g\rho}{\rho RT}$$

$$\frac{1}{\theta}\frac{\partial\theta}{\partial z} = \frac{1}{T}(-\gamma) + \frac{g}{C_p T}$$

or
$$\frac{T}{\theta}\frac{\partial\theta}{\partial z} = \gamma_d - \gamma$$

For dry or unsaturated air

$$\frac{\partial\theta}{\partial z} < 0 \quad \text{Unstable}$$

$$\frac{\partial\theta}{\partial z} = 0 \quad \text{Neutral}$$

$$\frac{\partial\theta}{\partial z} > 0 \quad \text{Stable}$$